J Econ Inequal
DOI 10.1007/s10888-006-9047-3

Editorial : Serge Kolm's "The Optimal Production of Social Justice"

Peter Lambert

Serge-Christophe Kolm has a claim dating back to the middle 1960s to some of the most fundamental theoretical constructions which underpin the modern study of income inequality. We are delighted to introduce the new "Rediscovered Classics" section of the *Journal of Economic Inequality* by reproducing here a key passage from the English-language text in which Serge's constructions are to be found. Serge's article "The optimal production of social justice" appears in the Proceedings of a 1965 Round Table Conference held by the International Economic Association in Biarritz, France, published in book form in 1968 in French and in 1969 in English in *Public Economics: An Analysis of Public Production and Consumption and their Relations to the Private Sectors*, edited by J. Margolis and H. Guitton, pages 145–200.

It is ironical, surely, that constructions of such fundamental importance went unnoticed by many in the late 1960s, trickling out in journal articles *by other authors* in the period from 1970 into the middle 1980s – and becoming known to generations of subsequent researchers under the names of these other authors. Thus, the *equally distributed equivalent income* (Serge's *equal equivalent income*) and *cost of inequality* concepts, along with the *Lorenz* and *generalized Lorenz dominance* theorems, which are generally attributed to Atkinson [6] and Shorrocks [11], can all be found in Serge's 1965 conference presentation. It is these constructions, more than anything else, which triggered the considerable wave of inequality research that began in the 1970s and continues to this day, leading ultimately to the foundation of the *Journal of Economic Inequality* itself.

We are delighted to reproduce here the relevant excerpt from Serge Kolm's "The optimal production of social justice" with the permission of Palgrave Macmillan, publisher of the 1969 English version, whom we thank. The paper was re-typeset and re-published in 2001 as Chapter 31 in M. Blaug (ed.) *The Foundations of 20th Century Economics : Landmark Papers in General Equilibrium, Social Choice and Welfare, selected by Kenneth J. Arrow and Gérard Debreu.* The UK publisher of this volume, Edward Elgar, kindly provided us with electronic copy from which the excerpts reproduced here were actually prepared.

The theory of inequalities in the strict sense is to be found in Sections 6 and 7 of Serge Kolm's paper, and it is these sections, along with some excerpts from the introductory sections, which we reproduce here. Serge writes "A concept of inequality is normative or is not. Hence, when we speak of inequality, we speak either of dispersion or of injustice. The Biarritz paper deals with ethical evaluation of the income distribution, hence its prime focus was upon unjust inequalities, and concepts which described the corresponding injustice or justice (which is more to the point)".

The equally distributed equivalent income and cost of inequality concepts, along with what has become widely known as "the Atkinson index" (though some, to their credit, call it the KAS or Kolm-Atkinson-Sen index), are to be found here - on pages 186-187 in Kolm [3], pages 635-636 in Kolm [4] and page A8. The index is illuminatingly named *relative injustice per dollar of social income*. The all-important Lorenz and generalized Lorenz dominance theorems can both be found here too - on page 193 in Kolm [3], page 640 in Kolm [4] and pages A15-A16 here. Serge refers to the dominance conditions as *constant-sum isophily* and *isophily* respectively. One must wonder why Serge did not get the credit for these path-breaking constructions at the time. Perhaps his penetrating and elegant deductive style, which is also very condensed, was difficult for economists to grasp?

Serge-Christophe Kolm grew up during the German occupation of France, when inequality was extreme, and in the immediate post-war era with its colonial problems. Against this background, of justice and injustice, Serge entered the Ecole Polytechnique in 1953 and chose to write his dissertation on income distribution and redistribution. His advisers at that time were François Divisia, a co-founder of the *Econometric Society* and of *Econometrica*, and Paul Lévy. Serge's studies concerned the effect of income transfers on the distribution, its concentration and Lorenz curves. He also considered the effect on social welfare, relating this to bistochastic matrices.

Serge's early researches extended work of Ostrowski [9] on Schur's [10] theorem and made a tentative application to the effects of income redistribution in France. In 1958 Serge chose to undertake practical work on development and, as director of the Sénégal Development Mission, found himself considering multidimensional inequality issues (in income, health and schooling) and taking measurements in the valley of the Sénégal river, more widely in the provinces of Sénégal, and ultimately in the various states of what was then French West Africa. He dryly notes "Statistics were fully insufficient at some places, and rather good in others… The endeavour had some purely conceptual aspects". He once spoke to me of a *concentration hyperdominance relation*, which he developed at that time in order to aid his assessment of the cultural fate of the peasants whose valley was to be flooded according to a plan of the colonial authorities. It would be fair to say of Serge Kolm at this point that he was "well ahead of his time".

Back in Paris, Serge presented the concepts and results he had built up at a 1961 international conference on development, and then with the help of the National Statistical Institute developed questionnaires to elicit information on people's feelings about inequality, in relation to mathematically equivalent (or related) properties. It was here that Serge coined the term "isophily" (liking equality) to characterize inequality aversion. In 1963, Serge was invited to Harvard by Wassili Leontief because of this work, and he stayed there for 3 years, working on income distribution and inequality, and also on the application of similar concepts to risky choices (this work was published in Serge's 1966 book in French, *Monetary and Financial Choices, Modern Theory and Technique*; the relation later to be called "second order stochastic dominance" and attributed to Hadar and Russell [7] and Hanoch and Levy [8] is presented and applied in this book). Serge's normative work on inequality at Harvard constitutes the stuff of the paper he presented at the 1965 conference on Public Economics held in Biarritz by the *International Economic Association*.

Serge's Biarritz paper contains a number of other, related sub-topics, in addition to the equally distributed (or equal) equivalent income, cost of inequality, Lorenz and generalized Lorenz dominance constructions which have become so fundamental to inequality measurement. In the theory of the moral or distributive surplus (sections 2 to 4), the philosophy is to derive the optimum distribution from people's moral preferences about the whole distribution (a case of "endogenous social choice"); the analysis of interdependent utilities and Pareto-improving redistribution are to be found there. There also is the theory of equivalence, in section 5, for choosing the incomes to compare, given people's differing labour supplies. For other expositions of the moral surplus, the theory of equivalence and endogenous social choice, see Serge's 2004 book *Macrojustice, The Political Economy of Fairness*.

Serge-Christophe Kolm's early work on inequality has an important place in the history of our subject, and is commended to you, the reader of *The Journal of Economic Inequality*, today.

Peter Lambert

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References

(a) Serge Kolm's cited publications

1. Kolm, S.-C.: Les Choix Financiers et Monétaires, Théorie et Technique Modernes. Dunod, Paris (1966)
2. Kolm, S.-C.: La production optimale de justice sociale. In: Margolis, J., Guitton, H. (eds.) Economie Publique, pp. 109-177. Editions du Centre National de la Recherche Scientifique, Paris (1968)
3. Kolm, S.-C.: The optimal production of social justice. Chapter 7, In: Margolis, J., Guitton, H. (eds.) Public Economics: An Analysis of Public Production and Consumption and their Relations to the Private Sectors, pp. 145-200. London: Macmillan (1969). Retypeset and republished as Kolm (2001)
4. Kolm, S.-C.: The optimal production of social justice. Chapter 31, In: Blaug, M. (ed.) The Foundations of 20th Century Economics: Landmark Papers in General Equilibrium, Social Choice and Welfare, selected by Kenneth J. Arrow and Gérard Debreu, pp. 606-649. Edward Elgar, Cheltenham UK and Northampton MA (2001)
5. Kolm, S.-C.: Macrojustice, The Political Economy of Fairness. Cambridge University Press, Cambridge, UK (2004)

(b) other cited works

6. Atkinson, A.B.: On the measurement of inequality. J. Econ. Theory **2**, 244-263 (1970)
7. Hadar, J., Russell, W.R.: Rules for ordering uncertain prospects. Am. Econ. Rev. **59**, 25-34 (1969)
8. Hanoch, G., Levy, H.: The efficiency analysis of choices involving risk. Rev. Econ. Stud. **38**, 335-346 (1969)
9. Ostrowski, A.: Sur quelques applications des functions convexes et concaves au sens de I. Schur. J. Math. Pures Appl. **31**, 253-292 (1952)
10. Schur, I.: Über eine Klasse von Mittelbeildunger mit Anwendungen die Determinanten: Theorie Sitzungsber. Berlin Matematische Gesellschaft **22**, 9-20 (1923)
11. Shorrocks, A.F.: Ranking income distributions. Economica **50**, 3-17 (1983)

The Optimal Production of Social Justice

S.-Ch. Kolm

I Excerpts from Introductory Sections

1 General Intent

(A) Useful Study of Public Economics Requires the Explicit Analysis of Social Justice

The distribution of welfare is one of the *raisons d'être* and fundamental social functions of the public economy. It is closely intermingled with the other one – the efficiency of the productive process in collaboration with the market – in all public economic problems: distribution of ownership and income, production of public goods and services, taxation, pricing of public utilities, market regulation, monetary measures; this tie appears both in the means and in the deep ends of growth, employment and price stabilisation policies.

For this reason, the help that economic policy may expect to receive from economics is limited by a serious shortcoming of the latter: descriptive economics has analysed at length both productive efficiency and income distribution; normative economics has many recommendations about how to achieve efficiency, but economics, in its present state, has almost nothing to say about the normative aspects of welfare distribution, i.e. about social justice*. Worse, welfare economists used to devote a great part of their endeavours to attempt to get rid of this problem, instead of trying to solve it as we shall do here.[1]

. . . .

**Note of 2006: Four decades later, one can only be struck by how drastically the situation has changed. Some would even say that we passed from the question* "*what can be said?*" *to the question* "*what remains to be said?*"*. This is the achievement of two generations of brilliant scholars in the fields of inequality and of normative economics.*

[1] It is difficult to argue that efficiency is more important than justice. Therefore one could say, loosely speaking, that more than nine-tenths of all economist-hours are devoted to the study of only half of the problem. Also, we hope we prove later that the marginal productivity of social justice analysis is not low. One of the possible reasons for this misallocation of resources may be that the United States is presently the leading country in economics, and it also happens to be the country in the world where, for clear historical and economic reasons, the population cares the least about distributive justice. Elsewhere, teaching economics is very much transforming justice-minded students into efficiency-minded economists. Furthermore, the British economic tradition, which the United States follows, has been less interested in this problem than most others, the Italian one in particular (see my comments on Professor Sen's paper in this volume).

(C) Theoretical and Practical Determination of the Social Optimum

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The definition of the optimal distribution of welfare does not result from any value judgement made by the economist. He is an observer of citizens' value judgements and opinions, as he is an observer of their tastes concerning consumers' goods. From these data he may deduce, in the same manner, the optimal production of goods and the optimal distribution of wealth. Useful normative economics is therefore a positive science since its basis is the *objective* observation of subjective opinions.

But the knowledge of these opinions presents exactly the same 'revelation' difficulties as that of the tastes for public goods.[2] As for any knowledge problem, the efficient method is the scientific one which consists of the interaction between observation procedures and an explicit theoretical construction. For our problem, the former are opinion polls, analysis of political phenomena (votes, selection of elites, etc.), studies of descriptive ethics, etc. We must propose bases for the latter, i.e., we must begin the construction of a theory of social justice. To compare with a neighbouring field, this theory will be to these observations what the theory of consumer's choice is to demand analysis.

A theory of social justice must begin by discovering the objective properties of the concepts under consideration. Speaking of *noumena*[3] a property is said to be objective if it is subjective for everybody; it may then be included in a definition of the vocable.[4] Now, as is well known by election candidates, consensus is the wider the less precisely defined the property is: collective will is 'the clearer the vaguer'. But, as they verify when elected, a property is the more useful to guide choices the more precise it is. Hence the first task of the science of social justice is to seek properties which both stand a good chance to be considered as 'natural' and are defined with precision, to show the field of precise and specific consensus.

The result of this research is that, in the question of social justice, informed opinions happen to present much wider domains of coincidence and consensus than is *a priori* believed. Among these is compensatory justice which is distinguished from distributive justice partly for this reason. But, chiefly, even

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[2] And, similarly too, but less than for some public goods, to the problem of the knowledge of opinions by the observer is added that of their own opinions by the individuals themselves. Paul Valéry said about this point, « politics has long been the art to prevent people from taking care of their own concern; it now consists in asking them questions on issues about which they have no idea ».

[3] That is, intelligible objects, as opposed to phenomena or sensible objects. Our previous argument is that somebody's noumenal conceptions are phenomena for an observer.

[4] The word « property » needs, of course, to be made more precise; on this point, see the difference between « moderate egalitarianism » and « satiation » which is mentioned in Part 7.

about distributive justice people agree more than they themselves think. The reason is that the logical equivalence of some properties is not visible without using a rather advanced mathematical apparatus in proofs which are sometimes far from elementary. As a result, some persons admit some properties and find no justification to others, other persons have the reverse opinion, whereas all these properties may turn out to be logically equivalent.[5]

(D) Sample of Problems Solved or in the Solution of which this Study Participates

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The relations between production and distribution originate in incentives to produce and in the distribution of capital (which may be tangible, human or social, on the one hand, received, innate or earned, on the other). They create the problems of choice between justice and efficiency, which are discussed in Parts 5 and 6 (and to the solution of which Part 7 contributes).... Income transfers and the choice of the optimal progressivity of income taxes will be among the favourite fields of application. But problems of distributive and compensatory justices appear in a non-negligible way in almost all questions of taxation, public expenditures, regulation, etc. The distributions under consideration may be among the individuals of a given society, among societies, and among generations.

The resulting optimum is not in general a situation of maximum social income (exhaustively reckoned, including consumption of leisure: see the 'leisurely equivalent' of Part 5). Indeed, societies often sacrifice production in order to achieve some justice.[6] Therefore, the usual measure of the welfare of a society, *per capita* social income, is not a good measure since it omits the 'consumption' of social justice. It is suggested in Part 6 that the good measure for a distribution of incomes out of which the effects of compensatory justice have been taken care of (cf. Part 5) is its 'equal equivalent,' i.e., the individual income which, if everybody had the same income, would yield a situation as good as the one under consideration; it is generally equal to the 'economic equivalent' of the distribution, which is the smallest *per capita* income of the situations equivalent to the one under consideration. The injustice of the distribution can be measured by the difference between the *per capita* income and this equivalent; or by one minus its justice measured by the ratio of the latter to the former – the welfare productivity of social income (a production of dollars of welfare per dollar of

[5] An important instance is given by the various ways of expressing that the Lorenz curve of a distribution is thoroughly « above » that of another. Cf. Parts 6 and 7.

[6] In some societies, though, social income would increase if ownership and income were less unequally distributed.

social income). The fundamental concepts and properties of judgements about distributive justice are presented in Part 6 and their logical relations are shown in Part 7 which we believe to be very important.[7]

. . . .

(3) VARIABLES OF DISTRIBUTIVE JUSTICE

Since compensatory justice deals with differences in the shape of indifference loci, distributive justice may reduce its variables to one utility index for each individual. Many specifications of these ordinal utility indexes could be chosen. However, one of these specifications is more operational and practically meaningful than the others: that is a money income. But the definition of the relevant income must free it from the effects taken care of by compensatory justice.

Now, the income of a given period, which is used in present and future consumption and in gifts and legacies,[8,9] stems from labour and capital, these two terms being understood in their broadest sense. In order to make income a variable of distributive justice, the first task of compensatory justice is to subtract from it the just compensation for labour.

*(D) Work Compensation: the Leisurely Equivalent**

(1) DEFINITION

Consider an individual. Let c be the vector of the quantities of goods he consumes or acquires. Let t be the vector of the parameters of his labour: durations, intensities, conditions, etc., of the various types of labour he performs. The parameter t_0 is always the total duration of his labour and $t_0=0$ means that he does not work. Let $u(c, t)$ be an ordinal utility index for this individual. When $t_0=0$, u is not sensitive to the other parameters t_i. The

[7] Parts 6 and 7 together may be read independently from the others.

[8] Of course, taxes are deducted and the money value to the individual of collective services are added (they cancel each other in case of 'benefit' taxes or tolls with perfect and complete discrimination according to both consumer and unit of the service).

[9] One of the reasons why distributive justice cares about disposable incomes and not about utility functions is the argument that 'each one is responsible for his own tastes', and therefore that the way in which people choose to lay out their income is their own concern; distributive justice cares only about the distribution of undifferentiated purchasing power, quite apart from its use. In fact, what is basically distributed is freedom of economic choice.

**Note of 2006 : From the theory of equivalence (Kolm 2004, chapter 26), the best income compensated for differences in labour is, rather than the leisurely equivalent income, the solution indicated in footnote 45 (note 12 in this document), with n being average labour.*

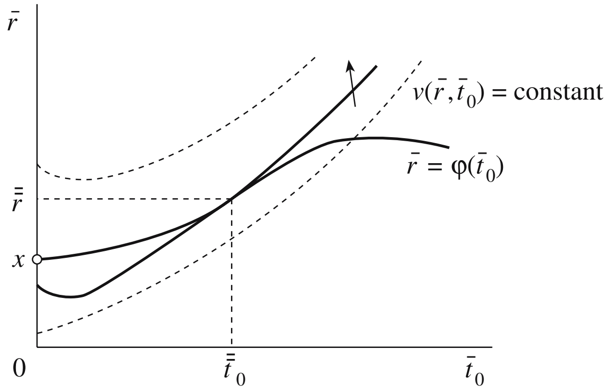

Figure 3

individual's physical and intellectual capacities are described by $t \in T(c)$. It may be that T depends little on c in wide ranges.[10] Let $r = f(t)$ be the individual's income. f reckons together incomes from all origins, including material capital (itself defined so as to include financial assets). This notation is used because some 'tangible' assets yield an income by co-operating with some elements of t, and although the income yielded by other such assets is independent of t. Finally, let $c \in b(r)$ be the individual's budgetary constraint. b depends on prices or on supply curves, and on non-negativities, discontinuities, and other physical constraints on c.[11]

Then, the *leisurely equivalent income* x is defined by the equation

$$\begin{array}{ll} \text{maximum of} & u(c,t) \\ \text{subject to} & \left| \begin{array}{l} c \in b(r) \\ r = f(t) \\ t \in T(c) \end{array} \right. \end{array} = \begin{array}{ll} \text{maximum of} & u(c,t) \\ \text{subject to} & \left| \begin{array}{l} t_0 = 0 \\ c \in b(x) \end{array} \right. \end{array}$$

x is income compensated for work disutility (and, eventually, pleasantness).[12]

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[10] Among possible reasons for this dependence are the effects of nutrition, housing, transportation and information services, etc.

[11] The *marketing work* necessary to buy c could also easily be taken into account.

[12] An 'n-hour equivalent' with $t_0 = n$ instead of $t_0 = 0$ could have been chosen (for instance, $n = 8$ for Oscar Lange in *On the Economic Theory of Socialism*), but in this case u would depend on t in the right-hand maximisation and the problem would be more complicated, whether the other components of t remain free or whether they are fixed too.

(2) GRAPHICAL REPRESENTATION AND EXAMPLE

x may be represented graphically (Fig. 3) by $v(\bar{r},\bar{t}_0) \triangleq \max\ u(c,t)$ subject to $c \in b(\bar{r}),\ \bar{r} = f(t),\ t \in T(c),\ t_0 = \bar{t}_0$, and by calling $\bar{r} = \varphi(\bar{t}_0)$ the solution of $\bar{r} = f[t^*(\bar{t}_0,\bar{r})]$ where t^* is the t that results from the maximisation. Then, the remaining problem has only two variables, $\bar{r}$ and $\bar{t}_0$ and the maximisation of $v(\bar{r},\bar{t}_0)$ subject to the constraint $\bar{r} = \varphi(\bar{t}_0)$ yields the optimum $\left(\bar{\bar{r}},\bar{\bar{t}}_0\right)$.

For instance, if labour income is independent from owned material capital and if the wage-rate s is fixed, $\varphi(\bar{t}_0) = r_K + s\bar{t}_0$ where r_K is the income yielded by this capital. If, moreover, $v(\bar{r},\bar{t}_0) = \bar{r} - \alpha\bar{t}_0^2 - \beta$, then $x = r_K + \frac{1}{2}s\bar{\bar{t}}_0 = \bar{\bar{r}} - \frac{1}{2}s\bar{\bar{t}}_0$: the leisurely equivalent income is capital income plus half labour income (or total income less half the labour income).

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II The Theory of Unjust Inequalities

6 Properties of Opinions of Distributive Justice

(A) Problem, Definitions, Notations

(1) PROBLEM

Traditional economics deals with efficiency analysis and gives, with utility theory, the tool for the analysis of compensatory justice. Hence, there remains to build the analysis of distributive justice. The general method of Part 2 is valid for this purpose. But there remains to know citizens' opinions about the justice of the distribution. A systematic approach to this vast problem must be based on an *a priori* analysis of the structures of the opinions about distributive justice. It has been remarked in Part 1 that a property which belongs to every citizen's opinion may be called 'objective', i.e., that it may be included in a definition of the terms 'justice' and 'just'. Hence, much hope and interest is derived from the results of the next Part which show many unexpected ties, implications and equivalences between very natural but seemingly unrelated properties. This Part is devoted to the exposition of these properties.

(2) DISTRIBUTIVE JUSTICE OPINIONS AND VARIABLES

Two precautions are necessary in order to deal with an opinion about distributive justice free from other effects. First, the opinion considered is not the one which would be expressed for tactical reasons in order to improve the situation of he who expresses it or to modify the situations of people he personally likes or dislikes; in brief, it is an impartial opinion. Such an opinion

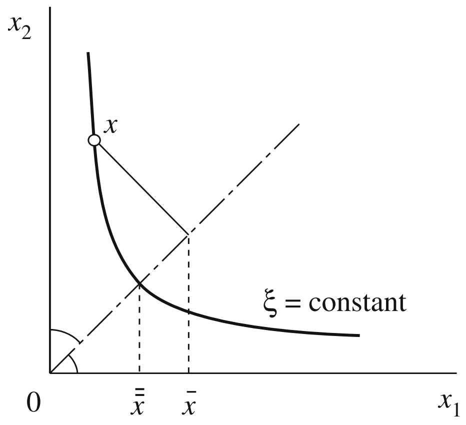

Figure 4

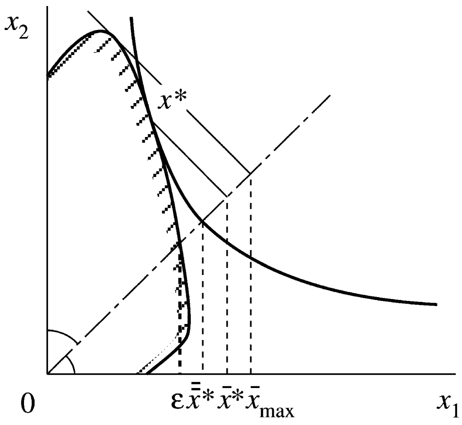

Figure 5

is expressed by a rational individual when indeed this expression does not influence these situations: in a mere investigation, by a vote in a political poll with a large number of voters,[13] when he does not belong to the society for which distribution is judged, etc. Second, the variables must be individual incomes x_i of each individual i, corrected for the effects of compensatory justice, as defined in the previous Part.

(3) PREFERENCES AND EQUAL EQUIVALENT

Let n be the number of individuals, x the vector of the x_i's, $X = \sum x_i$ the 'social income', $\bar{x} = \frac{\sum x_i}{n}$ the '*per capita*' or 'average' income. Following the usual terminology, let $\frac{x_i - \bar{x}}{\sigma}$, where σ is the standard deviation of the distribution of the x_i's, be the 'reduced income'. The distribution is *equal* if $x = e\bar{x}$ where e is the vector of n ones, and *unequal* in the opposite case. Let us represent the described opinion by rational preferences,[14] and let $\xi(x)$ be any specification of the corresponding ordinal index. Call $\xi_i = \frac{\partial \xi}{\partial x_i}$.

The *equal equivalent* of x is the scalar $\bar{\bar{x}}$ such that $e\bar{\bar{x}} \sim x$, that is, $\xi(e\bar{\bar{x}}) = \xi(x)$. Therefore $\bar{\bar{x}}$ is a ξ if ξ is an increasing function of $\bar{\bar{x}}$ at $e\bar{\bar{x}}$ (it is sufficient for this that, at this point, $\xi_i \geqq 0$ for all i, with the inequality holding for at least one i).

Figures 4 and 5 show the problem in the case n=2. $\varepsilon = \max\,[\bar{x} : x = e\bar{x}]$ for possible x's, is the equal maximum.

[13] Compare with perfectly competitive markets, where the acts of individually unimportant agents reveal the true values of the goods to them...

[14] For which we shall use the usual ordering notations $\succ$, $\sim$ and $\succsim$.

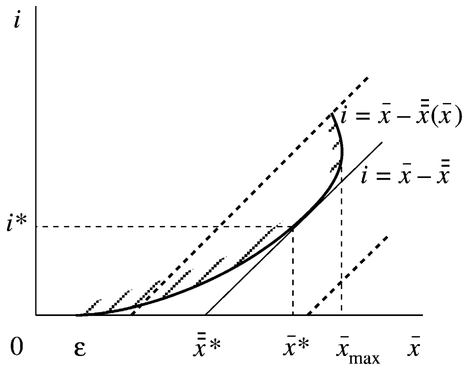

Figure 6

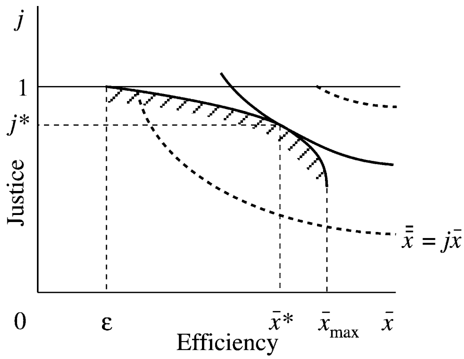

Figure 7

(4) JUSTICE AND INJUSTICES OF THE DISTRIBUTIONS

If $x = e\bar{x}, \bar{x} = \bar{\bar{x}}$. Hence, $i = \bar{x} - \bar{\bar{x}}$ is a monetary measure of the injustice of the distribution. Call it *injustice* and call $\hat{\imath} = \frac{i}{\bar{x}} = 1 - \frac{\bar{\bar{x}}}{\bar{x}}$ the *relative injustice*. i is injustice per person and $\hat{\imath}$ is injustice per dollar of social income. Then, $j = 1 - \hat{\imath} = \frac{\bar{\bar{x}}}{\bar{x}}$ will naturally be called *justice*. It is the productivity of social income, i.e., the efficiency of each dollar of social income, in welfare measured in money. $\frac{\bar{\bar{x}} - \bar{x}}{\sigma} = -\frac{i}{\sigma}$ will, of course, be called the reduced equal equivalent.

(5) ECONOMIC EQUIVALENT, RELATIVE COST AND YIELD OF THE DISTRIBUTION

Other meaningful measures are the *economic equivalent* of the distribution: $\tilde{x} = \min\,[\bar{x}^0 : x^0 \sim x]$ where x^0 is a variable distribution vector. The cost of the distribution is $c = \bar{x} - \tilde{x}$. The relative cost of the distribution is $\hat{c} = \frac{c}{\bar{x}} = 1 - \frac{\tilde{x}}{\bar{x}}$. And the yield of the distribution is $d = 1 - \hat{c} = \frac{\tilde{x}}{\bar{x}}$.

However, with the most interesting of the properties studied below, $\tilde{x} = \bar{\bar{x}}$, $c = i$, $\hat{c} = \hat{\imath}$, $d = j$.

(6) CHOICE BETWEEN JUSTICE AND EFFICIENCY

In order to single out the choice between justice and efficiency, let us first perform an efficiency suboptimisation maximising ξ on the set of possible distributions, given $\bar{x}$. The equal equivalent of the outcome is a function $\bar{\bar{x}}(\bar{x})$. The choice (generally a compromise) between justice and efficiency is then shown by Figures 6 and 7.

(7) COMPARISON OF REDISTRIBUTIONS

A transformation $x \rightarrow \phi(x)$ preserving X is an income *redistribution*. Given two redistributions ϕ^1 and ϕ^2, the latter is said to be uniformly preferred (alternatively, strictly uniformly preferred) to the former if, for all x, $\phi^2(x) \succsim \phi^1(x)$ (alternatively, $\phi^2(x) \succ \phi^1(x)$).

(B) Properties

The properties which appear to be natural and meaningful under analysis are the following. They all are ordinal. Numbers 1 and 2 denote two distributions and we use standard notations of vector inequality ($\geq$ means $\geqq$ for all dimensions and $>$ for at least one). These properties are for all x^1 and x^2 in the relevant domain.

Benevolence

benevolence : $\quad x^2 \geq x^1 \Rightarrow x^2 \succ x^1.$

non-malevolence : $\quad x^2 \geq x^1 \Rightarrow x^2 \succsim x^1.$

Impartiality or non-discrimination: x^π denoting the vector obtained from x by the permutation π of its co-ordinates ($\pi = 1, \ldots, n!$), $x^\pi \sim x$ for all π's.[15]

Rectifiance

rectifiance : $\quad (x_i - x_j)(\xi_i - \xi_j) \leqq 0 \quad$ for all i, j.

strict rectifiance : $\quad x_i < x_j \Rightarrow \xi_i > \xi_j \quad$ for all i, j.

Therefore, an opinion is rectifiant if, *ceteris paribus*, a dollar more to society is thought to be better if it goes to the poorer than to the richer of any two individuals, and a transfer of a dollar from the latter to the former is thought to be a good thing.[16]

[15] If the i's are different generations, a reconciliation between Ramsey's and the Sovietic time-preference on the one hand, and the other economists' on the other, is to say that preference is impartial but that capital productivity causes that, at the optimum, x_1 and x_2 being incomes at dates t_1 and t_2 such that $t_2 > t_1$, $\xi_2 < \xi_1$ and therefore the discount rate is positive (see Fig. 5).

[16] For instance, opinions expressed by Pigou (*Wealth and Welfare,* p. 24) and Dalton ('The Measurement of the Inequality of Incomes', *Economic Journal*, September 1920, p. 351) are strictly rectifiant; the latter calls rectifiance the 'principle of transfers'.

Isophily

Call $y_i = \min\limits_{\pi} \sum\limits_{j=1}^{i} x_j^{\pi}$ and $y = [y_i]$. Roughly speaking (since some x_j's may be equal) y_i is the sum of the i smallest x_j's. Note that y_n=X, $x^2 = x^1 \Rightarrow y^2 = y^1$, $x^2 \geq x^1 \Rightarrow y^2 \geq y^1$. The curve $\left(\frac{v}{n}, \frac{y_v}{y_n}\right)$ is the Lorenz curve of the distribution x.[17] The following properties are then defined:

constant-sum isophily : $\quad X^2 = X^1$ and $y^2 \geq y^1 \Rightarrow x^2 \succsim x^1$.

strict constant-sum isophily : $\quad X^2 = X^1$ and $y^2 \geq y^1 \Rightarrow x^2 \succ x^1$.

In these two properties, x^2's Lorenz curve is 'never under' x^1's.

super-isophily : $\quad X^2 > X^1$and $y^2 \geq y^1 \Rightarrow x^2 \succ x^1$.

isophily : $\quad y^2 \geq y^1 \Rightarrow x^2 \succsim x^1$.

strict isophily : $\quad y^2 \geq y^1 \Rightarrow x^2 \succ x^1$.

Averages preference

An opinion is said to prefer averages when it favours an income redistribution which, without changing the total sum, replaces each income by a linear

[17] ξ is a functional of the curve (v, y_v). $\bar{\bar{x}}$ is the slope of the straight line equivalent to this curve. For some problems it is useful to consider the characteristic numbers $\frac{v}{n}$ where v is, respectively, defined by $x_v = \bar{x}, x_v = \bar{\bar{x}}, y_v = v\bar{\bar{x}}$ (cf. Fig. 8).

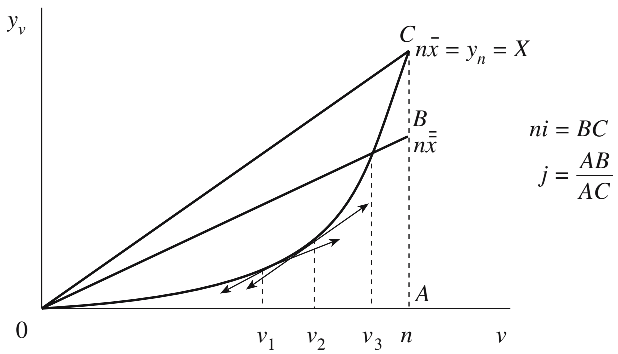

Figure 8

Note that for the first of these v's, $v\bar{x} - y_v = \frac{1}{2}\sum |x_i - \bar{x}|$.

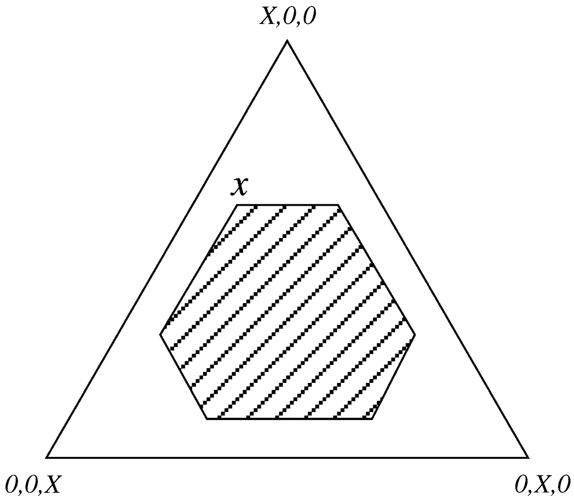

Figure 9

average (linear convex combination) of the former ones because this somewhat attenuates the injustice due to the inequality of the initial distribution. This redistribution transforms x^1 into x^2 such that, for all i, $x_i^2 = \sum_j b_{ij} x_j^1$ where $b_{ij} \geqq 0$ for all i, j and $\sum_j b_{ij} = 1$. Moreover, $X^2 = X^1$ implies $\sum_j \left(1 - \sum_i b_{ij}\right) x_j^1 = 0$, and since there exists at least n independent vectors x^1, $\sum_i b_{ij} = 1$ for all j. Therefore, the matrix $B = [b_{ij}]$ is bistochastic.

Note that if x^1 is an equal distribution, $x^2 = x^1$. Conversely, the reader should verify that Bx where B is bistochastic is the only linear transformation which (i) transforms every non-negative vector into a non-negative vector, (ii) preserves the total sum X, and (iii) transforms at least one equal distribution into an equal distribution.

Then, the following properties are defined

averages preference: $Bx \succsim x$.

strict averages preference : $Bx \succ x$ if $Bx \neq x^\pi$ for any π.

Call Bx a *linear vector average* and call the transformation $x \rightarrow Bx$ an *equalising redistribution*.

Mixtures preference

For an impartial opinion, $x^\pi \sim x$ for all π. A mixture by linear convex combination of the vectors x^π is in some sense more equally distributed without this changing the total sum of incomes. It may be preferred to x for this reason. Hence the following properties are defined for every set of non-negative numbers λ_π such that $\sum \lambda_\pi = 1$:

$$\textit{mixtures preference}: \quad \sum \lambda_\pi x^\pi \succsim x.$$

$$\textit{strict mixtures preference}: \quad \sum \lambda_\pi x^\pi \succ x \text{ if } \lambda_\pi \neq 1 \text{ for all } \pi.$$ [18]

Figure 9 shows the plane X=constant for the case n=3, an x, the $n! = 6\ x^\pi$, and the 'mixtures' preferred to x.

Satiation

satiation: the preference is quasi-concave, i.e., $\{x : x \succsim x^0\}$ is convex.
strict satiation: the preference is strictly quasi-concave, i.e., $\{x : x \succsim x^0\}$ is strictly convex.
constant-sum satiation: $\{x : x \succsim x^0,\ X = X^0\}$ is convex.
concavity or convexity of injustice: i and $\bar{\bar{x}}$ are functions of x; when one is concave or strictly concave, the other is convex or strictly convex, in the same domain.

Intensive, increasing, decreasing justice

The justice of a distribution is often felt as depending only on relative incomes $\frac{x_i}{x_j}$, i.e., as an intensive property; then, λ being a scalar, $j(\lambda x)=j(x)$; this is equivalent to $\bar{\bar{x}}$ and i being functions of x homogeneous of degree 1 and to indifference loci being homothetic with each other from the origin. Justice is said to be, respectively, increasing or decreasing according as, for $\lambda>1$, either $j(\lambda x)>j(x)$ or $j(\lambda x)<j(x)$ holds, which is equivalent to either $\bar{\bar{x}}(\lambda x) > \lambda \cdot \bar{\bar{x}}(x)$ or $\bar{\bar{x}}(\lambda x) < \lambda \cdot \bar{\bar{x}}(x)$, and to either $i(\lambda x) < \lambda \cdot i(x)$ or $i(\lambda x) > \lambda \cdot i(x)$.

[18] Mixtures preference suggests a method of redistribution: in order to transform x into $\sum \lambda_\pi x^\pi$, one may share all incomes in the same proportions (the λ_π's) and then permute each of the so-defined 'brackets' among individuals. This operation can be realised by exchanges between individuals taken two by two of their corresponding 'brackets', which is equivalent to an income transfer from the richest to the poorest of the two. Therefore, a mixture obeys Dalton's principle of successive transfers (op. cit. p. 351). The converse results from theorem 1 below.

If transfers are costly, one may want to minimise their number. To begin with, one may try to minimise the number of permutations, i.e., that of proportional 'brackets', i.e., that of $\lambda_\pi \neq 0$ (although this is not equivalent since in a permutation some individuals may correspond to themselves). From Caratheodory's theorem this number can always be reduced to n at most. It can often be further reduced.

Absolute, increasing, decreasing injustice

The injustice of a distribution is absolute if it does not change when all incomes vary by the same quantity, positive or negative, μ, i.e., when $i(x + e\mu) = i(x)$; this is equivalent to $\bar{\bar{x}}(x + e\mu) = \bar{\bar{x}}(x) + \mu$ and to the fact that indifference loci correspond to each other by translations parallel to e. Injustice is said to be, respectively, increasing or decreasing according as, for $\mu > 0$, either $i(x + e\mu) > i(x)$ or $i(x + e\mu) < i(x)$, which is equivalent to either $\bar{\bar{x}}(x + e\mu) < \bar{\bar{x}}(x) + \mu$ or $\bar{\bar{x}}(x + e\mu) > \bar{\bar{x}}(x) + \mu$.

Decreasing justice and increasing injustice are two (non-equivalent) ways of saying that justice per person is a luxury.[19]

Justice sensitivity

Given two opinions for which the justice of a distribution x is, respectively, $j^1(x)$ and $j^2(x)$, it is said: that the former is more sensitive to justice than the latter for distribution x if $j^1(x) < j^2(x)$, that the former is uniformly more sensitive to justice than the latter if $j^1(x) \leqq j^2(x)$ for all x, that the former is strictly uniformly more sensitive than the latter if $j^1(x) < j^2(x)$ for all unequal distributions.

Independence

Independence means that the opinion about income distributions to the individuals of a group does not depend on income distribution in the remainder of the society. More precisely, call x_J the vector of the x_i's for $i \in J$ and x_{-J} the vector of the x_i's for $i \notin J$, write $x = (x_J, x_{-J})$ and specify by indexes 1 and 2 some of these vectors. Then, the opinion is said to be independent when (with corresponding preferences and indifferences),

$$\left(x_J^1, x_{-J}^1\right) \gtrless \left(x_J^2, x_{-J}^1\right) \Leftrightarrow \left(x_J^1, x_{-J}^2\right) \gtrless \left(x_J^2, x_{-J}^2\right)$$

for all such x's and all J.[20]

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[19] Authors generally express their opinion by saying that '*ceteris paribus*, such type of modification of incomes augments or diminishes inequality', being understood that a decrease in inequality is a good thing and an increase in it a bad one. But they fail to make precise whether they are concerned with relative or absolute inequality. However, here is how we understand the following authors. For Taussig (*Principles of Economics*, 1939, p. 485), justice is intensive. For Dalton (op. cit), justice is decreasing. For Cannan (*Elementary Political Economy* 1953, p. 137), Loria (*La Sintesi Economica*, 1934, p. 369) and Dalton (op. cit), injustice is decreasing.

[20] There exists more restrictive sufficient conditions for this to hold, which require that the relation is verified for some specified sets of at least $n-1$ sets J.

It is easily shown that a necessary and sufficient condition of independence is that, for all i, j and $k \neq i, j$, the opinion on income distribution between i and j does not depend on k's income, i.e., that $\frac{\partial \ \xi_i}{\partial x_k \xi_j} = 0$.

Obviously, (1) the opinion is independent if there exists a specification of the ordinal index ξ of the form $\sum f_i(x_i)$ where f_i is a function of x_i, and (2) any linear function of this specification has the same form, i.e., this form is a cardinal property. Conversely, the reader can easily verify that, if the opinion is independent, then, (1) there exist such specifications, and (2) they all constitute a unique cardinal specification, i.e., they all are transformed from each other by increasing linear transformations.

If, moreover, the opinion is impartial, this form can be taken as $\sum f(x_i)$. Then, obviously, f is any function of a unique family each member of which is transformed into any other by an increasing linear transformation.[21] Therefore, such an opinion is thoroughly characterised by the function $\frac{f''}{f'}$ or $\check{\imath} = -\frac{1}{2}\frac{f''}{f'}$ since the *real* integration of the differential equation $f'' + 2\check{\imath}f' = 0$ yields such a family. The reader can easily verify that $\check{\imath} = \lim_{\sigma \to 0} \frac{i}{\sigma^2}$: it is, for low-dispersion distributions, the marginal injustice per person per unit of variance of incomes. Hence we call it *marginal injustice*. The marginal injustice per dollar per unit of variance of distributed dollars, or, for short, the *marginal relative injustice*, is then $\lim_{\sigma \to 0} \frac{\hat{\imath}}{\sigma^2 / \bar{x}^2}$, i.e., w being the variable, $w \cdot \check{\imath}(\mathrm{w})$.

7 Fundamental Structure of Distributive Justice

We present in this Part the main theorems of distributive justice. For lack of space, we do not give the demonstrations. From a mathematical point of view, some of these relations are new; others are well known from mathematicians but the grouping of properties we present often enables one to give demonstrations much simpler than those which already exist.[22]

[21] That is the reason why we reproach Dalton (op. cit.) for choosing measures of the inequality of welfare distribution of the form $\frac{f(\bar{x})}{f(\bar{\bar{x}})}$ which depends on the addition of a constant to f.

Besides, let us notice that $\bar{\bar{x}}$, $\hat{\imath}$ and j satisfy this author's 'principle of proportionate additions to persons' since, calling n_k the number of individuals who have the same income x_k, $\bar{\bar{x}}$ defined by $f(\bar{\bar{x}}) = \sum \frac{n_k}{n} f(x_k)$, and $\bar{x} = \sum \frac{n_k}{n} x_k$, do not change when all n_k's (the sum of which is n) vary in the same proportion.

[22] That is in particular the case for the theorems of Birkhoff–von Neumann, Ostrowski, Hardy–Littlewood–Pólya and Schur.

(A) Moderate Egalitarianism

(1) FUNDAMENTAL THEOREMS

Theorem 1. The four following properties are equivalent: (i) rectifiance and impartiality together, (ii) constant-sum isophily, (iii) averages preference, (iv) mixtures preference.

Theorem 2. The three following properties are equivalent: (i) strict constant-sum isophily, (ii) strict averages preference, (iii) strict mixtures preference.

Theorem 3. Strict rectifiance and impartiality together imply the properties of theorem 2.

Remark. The reciprocal of theorem 3 is not true.

Theorem 4. The properties of theorem 1 and benevolence together are equivalent to super-isophily.

Theorem 5. The properties of theorem 2 and benevolence together are equivalent to strict isophily.

Theorem 6. The properties of theorem 1 and non-malevolence together are equivalent to isophily.

(2) COMPARISON OF REDISTRIBUTIONS

Theorem 7. Given two equalising redistributions whose matrices are, respectively, B^1 and B^2, either if the latter is uniformly preferred to the former for all opinions having the properties of theorem 1, or, alternatively, if the latter is strictly uniformly preferred to the former for all opinions having the properties of theorem 2, and if B^1 is regular, then there exists a bistochastic matrix B such that $B^2=BB^1$.

Remark. Theorem 7 states, of course, that B does not depend on x.[23] The latter redistribution is then said to be *more equalising* than the former. The obtained relation expresses that each row-vector of B^2 is a linear convex combination of the row-vectors of B.

[23] The proof of this theorem requires some negative x_i's to be considered, which, according to remarks of Parts 3 and 5, is acceptable and represents a debt or a tax on labour income.

(3) INDEPENDENCE

Theorem 8. For a benevolent, impartial and independent opinion, the properties of each of the two following groups are equivalent: (1) isophily, rectifiance, satiation, $i \geqq 0$, $\breve{\imath} \geqq 0$, concavity of f; (2) strict isophily, strict rectifiance, strict satiation, $i > 0$ out of the locus of equal distributions, $\breve{\imath} > 0$ except on a set of measure zero, strict concavity of f.

Theorem 9. (1) A distribution is not worse than another for all isophile opinions if and only if it is not worse for all independent, impartial, non-malevolent and rectifiant opinions. (2) A distribution is better than another for all strictly isophile opinions if and only if it is better for all independent, impartial, benevolent and strictly rectifiant opinions.

Remark. Of course, what is interesting is the sufficiency with independent opinions only.

(B) Intensive Justice and Absolute Injustice

(1) INTENSIVE JUSTICE AND ABSOLUTE INJUSTICE TOGETHER

Theorem 10. An opinion feels altogether justice to be intensive and injustice to be absolute if and only if the reduced equal equivalent depends on reduced incomes only, or equivalently, injustice is a linearly homogeneous function of incomes deviations from their mean.

Remark. Equivalently, injustice is the product of the standard deviation of the distribution by a function of reduced incomes only.

Theorem 11. If an opinion having the properties of theorem 10 either is independent or judges only two incomes, the ordinal preference index has a piecemeal linear specification and the equal equivalent and injustice are linear combinations of incomes.

Remark. The equal equivalent is a convex linear combination if, moreover, the opinion is non-malevolent.

Theorem 12. If an opinion having the properties of theorem 11 is impartial with a differenciable preference index, injustice is null and the equal equivalent is the *per capita* income.

(2) INDEPENDENCE AND INTENSIVE JUSTICE OR ABSOLUTE INJUSTICE

Theorem 13. Justice is intensive for an independent opinion if and only if the ordinal preference index has a specification of the form $\sum \alpha_i x_i^{\beta}$ or $\prod x_i^{\alpha_i}$ where β and the α_i's are constants.

Theorem 14. Injustice is absolute for an independent opinion if and only if the ordinal preference index has a specification of the form $\sum \alpha_i e^{\beta x_i}$ where β and the α_i*'s* are constants.

Note. Theorems 15 and 16 are corollaries of theorems 13 and 14 obtained when the opinion is impartial and hence when the ordinal preference index has a specification of the form $\sum f(x_i)$.

Theorem 15. Justice is intensive for an impartial independent opinion if and only if *f* has a power or logarithmic specification.[24]

Theorem 16. Injustice is absolute for an impartial independent opinion if and only if *f* has an exponential specification.

Remark. Theorems 11 and 12 may be obtained, respectively, from the conjunction of the properties of theorems 13 and 14 and of theorems 15 and 16.

Theorem 17. Non-malevolence and isophily hold if and only if: (1) for the properties of theorem 13, (a) for $\sum \alpha_i x_i^{\beta}$ either $\beta \leqq 0$ and $\alpha_i \leqq 0$ for all *i,* or $0 \leqq \beta \leqq 1$ and $\alpha_i \geqq 0$ for all *i,* and (b) for $\prod x_i^{\alpha_i}$, $\alpha_i \geqq 0$ for all *i*; (2) for the properties of theorem 14, $\beta \leqq 0$ and $\alpha_i \leqq 0$ for all *i.*

(C) Marginal Injustices for Impartial Independent Opinions

Theorem 18. Properties of theorems 15 and 16 are respectively equivalent to the constancy of relative marginal injustice and of marginal injustice.

Theorem 19. (1) For a benevolent, impartial, independent and isophile opinion, marginal injustice is convex (vs. strictly convex) if injustice is convex (vs. strictly convex out of the locus of equal distributions). (2) For a benevolent, impartial, independent opinion, injustice is concave (vs. strictly concave out of the locus of equal distributions) if $\frac{1}{i}$ is convex (vs. strictly convex).

[24] In this latter case, a specification of the ordinal index is $\sum \log x_i$. Therefore, this case may result from the conjunction of independence, impartiality, and an hypothesis similar to Bernoulli's on 'moral wealth' or to Weber-Fechner's on 'sensation'. It suggests that a Nation's welfare would be better measured by the *geometric mean* of incomes rather than by their arithmetic mean, or *per capita* income, which is usually chosen: this would make allowance for inequalities in distribution. Note incidentally that if income distribution is Log-normal, as has often been suggested (Gibrat, Champernowne, Brown and Aitchison, etc.), this suggestion amounts to using the median of incomes rather than their (arithmetic) mean; (in this case, the Lorenz curve is symmetrical).

Theorem 20. Given two benevolent, impartial and independent opinions marked by indexes 1 and 2, and for an interval of variation of incomes, (1) 1 is uniformly more sensitive to justice than 2 if and only if $\check{\iota}_1 \geqq \check{\iota}_2$ (or the function $f_2 f_1^{-1}$ is convex) in this interval; (2) 1 is strictly uniformly more sensitive to justice than 2 if and only if $\check{\iota}_1 > \check{\iota}_2$ with the possible exception of a set of measure zero (or the function $f_2 f_1^{-1}$ is strictly convex) in this interval.

Remark. Some properties of theorem 8 are corollaries of theorem 20 obtained when f_2 is linear and $\check{\iota}_2 = 0$.

(D) Aggregations of Societies, Distributions and Opinions

(1) AGGREGATION OF SOCIETIES

Theorem 21. Consider the gathering of several societies into a single one. Populations, social incomes, equal equivalents, justices, injustices, and relative injustices for the constituent societies represented by index k and for the aggregate society are respectively n_k and $n = \sum n_k$, X_k and $X = \sum X_k$, $\bar{\bar{x}}_k$ and $\bar{\bar{x}}$, j_k and j, i_k and i, $\hat{\iota}_k$ and $\hat{\iota}$. Then, for an independent, impartial, benevolent and rectifiant opinion,

$$\bar{\bar{x}} \leqq \sum \frac{n_k}{n} \bar{\bar{x}}_k,\ j \leqq \sum \frac{X_k}{X} j_k,\ i \geqq \sum \frac{n_k}{n} i_k, \hat{\iota} \geqq \sum \frac{X_k}{X} \hat{\iota}_k.$$

If, moreover, rectifiance is strict, the inequalities are strict if and only if all $\bar{\bar{x}}_k$s are not equal.

(2) COMPOSITION OF DISTRIBUTIONS

Theorem 22. If intensive justice and satiation hold, then, for distributions x^k,

$$\bar{\bar{x}}\left(\sum x^k\right) \geqq \sum \bar{\bar{x}}(x^k),\ i\left(\sum x^k\right) \leqq \sum i(x^k),$$

$$j\left(\sum x^k\right) \geqq \sum \frac{\bar{x}^k}{\sum \bar{x}^k} j(x^k),\ \hat{\iota}\left(\sum x^k\right) \leqq \sum \frac{\bar{x}^k}{\sum \bar{x}^k} \hat{\iota}(x^k).$$

If, moreover, satiation is strict, all inequalities are strict if and only if all x^k's are not colinear.

(3) AGGREGATION OF OPINIONS

Following Pareto, let us aggregate citizens' opinions represented by ordinal indexes $\xi^h(x)$ into a social opinion represented by the ordinal index

$\xi(x)$ thanks to a function F such that $\xi(x) \equiv F\left(\left[\xi^h\right]\right)$. The aggregation is said to be benevolent toward h if $F'_h > 0$. It is said to be non-malevolent if $F'_h \geqq 0$ for all h.

Theorem 23. (1) If all citizens are isophile and if the aggregation is non-malevolent, society is isophile. (2) If, moreover, there exists at least one strictly isophile citizen for whom the aggregation is benevolent, society is strictly isophile.

Remark. Hence one can say, roughly, that the properties of moderate egalitarianism are conserved by a benevolent aggregation. It is not so for the usually considered property, namely satiation; for it, the more general conservation theorem is *a priori* rather arbitrary: $\xi(x)$ is quasi-concave if there exists a set of convex specifications of the ordinal indexes $\xi^h(x)$ such that $F([\xi^h])$ be non-decreasing quasi-concave. In brief, moderate egalitarianism possesses together two complementary characteristics: people agree more than they think about it, and when they agree it must be a property of social preferences.

. . . .