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FREEDOM JUSTICE

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Abstract

The most general and central principle of social and economic optimality and justice is shown to be equal freedom. The standard and central case is that of freedom valued for the choice it permits. Allocations abiding by this principle are characterized, with the main structures of constraints and possibilities and the main alternatives as regards the corresponding entitlements and accountabilities. When such first best equal freedom is not possible or cannot be efficient or in the core, second-best freedom egalitarian principles are defined, notably in the category of freedom maximins. These solutions rest on the properties of freedom comparisons and of freedom-ordered allocations.

1. PHILOSOPHY AND OVERVIEW

1.1. The basic social ethic

"Men are free and equal in rights": This opening statement of the revolutionary Declarations of over two centuries ago constitutes the basis of the social ethics of modernity. This was meant for all persons and basic rights, but, more generally, equal freedom is the rationally necessary form of most principles of justice adequately conceived (apparently different principles are in fact limiting cases of this one). This will shortly be explained, but this result explains why it is useful to characterize equal freedom, and to define second-best freedom egalitarianism such as maximin in liberty when this equality is not possible or cannot be efficient B and efficiency will also be defined and vindicated in terms of liberty. This paper will provide the corresponding basic concepts, characterizations and properties in focussing on the most basic and general value of freedom as means of acts and choices.

The social ethic analyzed here will choose the most widely relevant option for each of the choices raised by the constitution of a social ethic. It can thus be seen, in this sense, as the central or most important social ethic. But the basic concepts of the metaethics of justice are essential and should be recalled to begin with[1].

The concept of justice can be derived from the more general concept of social optimality and of the social optimum, that is, the definition of the best in questions concerning society. Justice is the aspect of the social optimum which considers situations of social entities called the "justiciables", when the considered situations are "for the sake" of the justiciables. Justiciables can a priori be many things (such as individuals, families, nations, firms, groups defined according to any possible criterion, cultures, and so on). In *individualistic* justice, the justiciables are individuals. The expression "for the sake" may refer to justiciables' view or to an outside ("paternalistic") opinion. In the former case, the considered justice is *respectful*. Our concern here will be with respectful individualistic justice.

A very important issue about justice is the place of the considered items in the ethical judgment. The aspects of the justiciables' situations explicitly considered in a judgement of justice are called the *situational variables*. A conception of justice *ultimately* cares for certain items about which its evaluation is *direct*. They are the *directly* (or ultimately) *relevant* (or morally relevant) or evaluated items for this conception of justice. Other items evaluated by this conception are only indirectly morally relevant for it, and their moral evaluation is indirect and derived from that of the directly morally relevant items. These directly morally relevant items may be aspects of the justiciables, but they may also be other aspects of society (for instance, global aspects). The considered situational variables may be these directly morally relevant items, and the judgment concerning them then is *direct justice*. But they may also be other items, different from the directly morally relevant items though related to them, and the judgment concerning these situational variables is *indirect* or *derived justice*. For

[1] More complete presentations are proposed in Kolm 1990 and 1996a.

instance, we will consider the derived justice of individuals' allocations for a conception of justice which takes individuals' freedom of choice of these allocations as the variables directly relevant for justice. Then, the just allocations will be defined as those which result from a just distribution of individuals' domains of free choice[2].

In *respectful* individualistic justice, the items directly relevant for justice are individuals' means or ends. The means can be freedoms, rights, powers, assets, capacities, possibilities, etc. They can be seen as freedom in a broad sense of the term, and they determine a domain of free choice. We will consider here this freedom proper B rather than the limiting cases in which consumption goods are means for consumption, and consumption goods and capacities for being satisfied are means for satisfaction.

Taking this freedom as the item directly morally relevant for justice is the normal and most general case. Indeed, what the individuals do with their possibilities is generally considered a *private* issue, irrelevant to justice, distribution, and public policy (if it does not affect other people without their will). Individuals are prima facie responsible for their choices and acts, given their means, and hence they are normally held accountable for the transformation of their possibility sets into the actually chosen items. And the satisfaction, pleasure or happiness they derive from given means or consumption are most often considered a private issue irrelevant for justice which is a public and interpersonal concern. There are valid exceptions to this position, but they solely concern a minority of cases. Hence, we will focus here on this most general case of respectful individualistic justice taking individuals' freedom of choice as the items directly morally relevant for justice. Justice taking freedoms as the items it directly morally evaluates is freedom justice, or *eleutheristic justice* (from the Greek word for freedom, and in opposition to *eudemonistic justice* which takes individuals' happiness as the items directly morally relevant for justice)[3].

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[2] In addition to these two kinds of variables – the directly morally relevant items and the situational variables – there may be a third kind, the *instrumental variables*, which are the items actually manipulated in order to achieve the optimum and justice. These variables may be any of the other kinds.

[3] From a previous remark, one can also see eudemonistic justice as a limiting case of eleutheristic justice.

The classical model of individuals choosing in domains of choice according to their preferences so as to best be satisfied, will be retained here. Then, the irrelevance, for *direct* justice, of choice and of satisfaction amounts to this irrelevance of individuals' preferences which direct choice (and provide satisfaction). Hence, freedom is described, for this judgment, by the corresponding domain of choice[4].

The basic theorem of the metaethical theory of justice is that rationality in its most basic sense of "for a reason" implies that justice requires "prima facie identical treatment of justiciables identical in the relevant characteristics"[5]. In the present case, for direct justice, the objects of the "treatment" are the individuals' domains of choice, and the individuals otherwise solely differ by their given preferences, which are irrelevant. Hence, justice is, prima facie, identity of the domains of choice. "Prima facie" means in the absence of an overriding reason, which can be impossibility, or impossibility of this equality along with the satisfaction of any other criterion that may be relevant. Such a criterion can notably be nondomination by unanimous improvement in the relevant individual items. When there is such an overriding reason, principles of relevant second-best egalitarianism have to be defined B here principles of second-best freedom egalitarianism. Concepts of more or less free will then be needed, and the irrelevance of preferences entails that these concepts can solely be defined from inclusions of domains of choice.

An individual allocation will denote a bundle of consumption goods, plus possibly occupation, labor or leisure, and any other relevant aspect of the individual's situation. An allocation will a priori denote a set of individuals' allocations, one for each individual (when there is no risk of ambiguity, it may also stand for an individual allocation). We will classically consider that the individuals may choose their individual allocations in their domains of choice. The derived first-best eleutheristic justice (freedom justice) of allocations consists of allocations which can result from identical individual domains of choice. Similarly, concepts of higher or lower freedom will translate into the field of allocations as allocations which can result from set-included domains of choice. Hence, individuals'

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[4] That is, we do not consider, here, all the subtle aspects and types of freedom involving "mental freedom" (see Kolm 1996a, chap. 2).

[5] The most complete derivation of this result is to be found in Kolm 1998 (translation of 1971), foreword 1997, section 5 (see also 1990, 1993).

preferences are absent from direct freedom justice, but they may appear in the characterizations of indirect freedom justice concerned with allocations, since a chosen allocation depends on both the domains of choice and the preferences of individuals.

The most standard model of individuals will be kept here, because it sufficiently describes a large number of cases (and it can be defended for the present social ethical use on the ground of rationality). Other assumptions are studied elsewhere[6]. The individuals will be concerned with their domains of choice solely for what they can have with them. Choosing entails no intrinsic appreciation, cost, or anguish of choice, or preference or aversion for responsibility. Freedom is solely *instrumental* in this sense[7]. This is the most basic and general value of freedom, and a sufficient hypothesis in many or most cases. Hence, individuals are indifferent between being provided with either directly their allocations or any domain of choice in which these allocations are their best choices. Moreover, individuals' preferences are neither directly concerned with nor influenced by others' allocations or domains of choice (no externalities).

Let us also recall that one of the most classical methods in social ethics for judging situations consists of considering whether they could or would result from adequately characterized free choices or processes. Compensations for past violations of rights in law or in process liberal theory belong to this category. But the main theoretical example is that of the theories of social contracts, which have been basic in social ethics for the last four centuries. These methods belong to eleutheristic social ethics based on free choices and freedoms which need solely be potential, notional, hypothetical, or putative[8].

Allocations will therefore be evaluated according as they can be obtained from identical individual domains of choice, or from individual domains of choice which are one included in the other. When an allocation has such a property, it is possible to provide the

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6 For instance, for the intrinsic value of freedom, in Kolm 1982, 1993, and for preference externalities in relation with justice and equal freedom in 1966, 1991b, 1995.

7 The varied and numerous other possible values of freedom are analyzed in Kolm 1996a, chap.2. Other purely instrumental values of freedom can for instance be found when the domain of choice per se provides social status (which can entail other advantages).

8 The only type of social contract that rests on actual freedom is that which rests on consent (from Plato's *Crito* on). But this is a very particular and very dubious type of social contract (emigration or rebellion can be very costly) – see Kolm 1985, 1996a.

individuals with domains of choice having these relations and from which they choose the considered allocation. Yet, from the assumptions the individuals are indifferent between being provided with such domains of choice or directly with the allocation. Moreover, as just noted, for a classical, common, and widespread – though not necessary – view in social ethic, the mere possibility that the allocation can be obtained by such free choices of the individuals suffices. Hence, the individuals can actually be provided with corresponding domains of choice (or not), but solely the possibility will be the object of study here. We will thus analyze possible or potential freedoms corresponding to given allocations.

1.2. An outline of concepts and results.

1.2.1 Equal freedom, no less free, less free, freer.

The basic conceptual tool will be the sets of domains of choice which would or could lead to the considered allocation. An allocation is *equal-freedom* when it can be obtained from identical individual domains of choice. This turns out to occur if and only if there also is equal-freedom for all subsets, or solely for all pairs, of individuals. In fact, equal-freedom allocations turn out to amount to the situation where no individual prefers any other's allocation to her own. This constitutes, indeed, the basic and most important reason for the interest of this principle ("equity" for *equ*al *i*nstrumental *i*ndependant liber*ty*)[9].

An individual with her individual allocation is (potentially) *no less free* than another with her individual allocation if these allocations can be obtained from the choices of these individuals in domains such that the former includes the latter. It turns out that *as free as* (equal-freedom for the pair) amounts to each being no less free than the other. *Less free* is defined as no *no less free*, and *freer* as no less free and not equally free (or no less free in one sense and less free in the other).

Applied to different allocations of the same individual, these potential freedom comparisons amount to standard preferences: the individual prefers to be freer, does not prefer to be less free and is indifferent to being as free. Hence, Pareto efficiency is equivalent when expressed in terms of potential freedom or in terms of preferences. Another relevant concept is that of the core: if groups of individuals have the right or the power to redistribute their allocations among themselves, then, if a group can do this in benefitting all its members, the allocation either is actually unstable or it does not implement the possibilities of free action, agreement, or exchange[10]. The absence of such groups (from one individual to all) characterizes allocations in the core.

1.2.2. Indexes of freedom inequality, freest and least free.

The numbers or proportions of the noted pairwise relations provide various measures of the degree of freedom-equality and freedom-inequality of an allocation. They also provide indices of individuals' relative overall freedoms and unfreedoms, and freedom-rankings of individuals, with globally freest and least free, globally second freest and second least free, etc. For instance, such an index can be the number of individuals less free than a given individual minus the number of individuals to whom she is less free, or the converse, or similar differences with relations no less free B which amount to the same B, or with relations freer, or the index can be the number or fraction of individuals with whom she is or is not equally free. Then, second-best freedom egalitarian principles can be found in allocations which minimize these overall freedom inequality indices, or maximize the freedom of the least free with possible leximin extensions of these maximins, in the relevant domain such as possible, efficient, or core allocations.

1.2.3. Freedom-ordered allocations.

An allocation is *freedom-ordered* when the individuals can be ranked in such a way that each is no less free than the following ones. This is shown to be equivalent to the possibility of obtaining the allocation by individual choices in domains successively included into the preceding one(s). If the number of individuals is finite, an allocation is freedom-ordered if and only if there is no cycle (closed chain) of successive relations "less free". And

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[9] This will shortly be further discussed, along with the freedom-relevant variants of this principle.

[10] A public allocation may have the moral duty to implement the outcome of such free actions or agreements as a result of a "liberal social contract" (see Kolm 1985, 1996a).

if the permutations of individual allocations among the individuals are possible, the existence of such a cycle implies that of a permutation which makes all the concerned individuals freer (and better off) – an “improving permutation”. This implies that the allocation is not efficient and is not in the core. Hence, with a finite number of individuals and possible permutations (that is, a symmetrical possibility set), efficient (and core) allocations are freedom-ordered. Freedom-ordered allocations, and all their restrictions to subpopulations, have sets of equally free *freest* individuals who are no less free than all individuals, of equally free *least free* individuals, such that all individuals are no less free then them, and, if the allocation is not equal-freedom, of individuals in each category who are not in the other (*strictly least free* and *strictly freest*). *Minimally least free* individuals are least free with the largest number of individuals freer than them, and *maximally freest* individuals are freest with the largest number of individuals to whom they are freer. These properties provide the basis for various concepts of maximins and leximins shortly to be described.

1.2.4. Entitlements and accountabilities for personal possibilities and limitations.

Moreover, individuals may not be able to have each individual allocation, and the sets of the allocations each can have may differ. One important cause of such limitations occurs when allocations include occupations, or income or consumption goods that can be obtained with a given work, because individuals’ abilities and productive capacities differ. But needs and various social reasons can also cause such limitations and differences. These limitations and differences are amenable to two kinds of ethical treatment. In one case, the individuals are accountable for their limitations and entitled to their possibilities (or accountable for or entitled to the particular specificities of their own limitations and possibilities, that is, in the measure in which they differ from others’). This is, for instance, the case of self-ownership of classical process liberalism. Or individuals may be accountable for certain of their needs (other individuals have no duty to pay for the satisfaction of these needs). In the alternative case, these differences in possibilities and in limitations or handicaps are considered an injustice which should be corrected or compensated for by the considered public policy. It may also be that limitations and differences with different causes are treated differently in this respect.

These two different ethical judgments about individual possibilities and their differences entail two different technical treatments. When the policy is directly concerned with these differences in possibilities and tries to correct them, its notions abide by these constraints, that is, the potential domains of choice it considers for each individual solely are of individual allocations that this individual can have. By contrast, when the individuals are deemed accountable for their limitations or entitled to their possibilities, the potential domains of choice considered by the theory can contain individual allocations that the individuals cannot have, since the individuals are accountable for not choosing them. The ethic, in this case, is not directly concerned with these limitations. But the individuals will choose, actually or notionally, solely allocations which they can have. Hence, these possibilities will appear in the indirect expression of freedom justice which considers allocations, as it is the case with preferences (by contrast, in the previous case possibilities appear in the definition of direct freedom justice). Yet, possibilities and limitations are here morally left to the individuals, as their preferences are. In fact, this case can be reduced to that in which preferences only are considered, in introducing derived preferences in which an individual allocation that an individual cannot have is considered as one which she finds less good than all those she can have (the “sour grapes preferences” derived from actual preferences and possibilities).

In all cases, potentially equally free and no less free individuals with given allocations are still defined as the possibility that they choose their allocations from identical or included domains of choice. And it turns out that two individuals are equally free if and only if each is no less free than the other; and that a number of individuals are equally free if and only if this is the case for all subgroups, and even solely for all pairs.

There thus are three cases according as individuals: (1) have the same possibilities, and (2) are not or (3) are accountable for their differences. Then, equal freedom will be shown to respectively amount to: (1) no individual prefers any other’s allocation to her own; (2) the same plus “and each individual can have each other’s allocation”; (3) “no individual prefers, to her allocation, an other’s allocation that she can have”, or “each individual either does not prefer or cannot have each other’s allocation” (this was called “realistic equity”: individuals compare their allocation solely with those of others that they can have)[11]. Relatedly, potentially "no less free" will turn out to respectively amount to: (1) the individual does not prefer the other individual's allocation to her own; (2) she does not prefer it and she can have it; (3) she does not prefer it if she can have it. And potentially "less free", the opposite of "no less free", thus respectively amounts to: (1) prefers the other's allocation; (2) prefers it if she can have it; (3) prefers it and can have it. Finally, "freer", defined as no less free and not equally free, is no less free in one sense and less free in the other.

When applied to the same individual and different individual allocations that she can have, all these (potential) freedom comparaisons amount to preferences – that is, an individual prefers to be freer, and not to become less free, and is indifferent to being as free. One consequence is that Pareto efficiency amounts to the same when expressed in terms of comparison of potential freedoms or in terms of preferences as it standardly is (we will just say "efficiency").

The definition of freedom-ordered allocations from the relation no less free, and their general properties, are the same in all three cases. This includes the equivalence with the absence of "less free" cycles, with finitely many individuals. But, if the constraints other than individual possibilities allow for permutations, transferring an individual allocation to a less free individual is always possible if "less free" implies that she can have it (cases 1 and 3), but it may not be possible in the other case (case 2). Hence, the impossibility of such improving permutations implies that of less free cycles solely in cases 1 and 3, since in the other case a transfer to a less free may not be possible. Thus, efficient and core allocations necessarily are freedom-ordered, with finitely many individuals and otherwise possible permutations, in cases 1 and 3 only, that is, when individuals are entitled to or accountable for the differences in their individual possibilities (including the case where there is no such differences).

1.2.5. Maximins in liberty.

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[11] See Kolm 1971 (English translation, 1997), and applications in Kolm 1991a and 1993.

There may be no possible allocations that are equal-freedom, or equal-freedom and efficient, or equal-freedom and in the core. This is bound to result from limitations in divisibility or in transferability (for physical or, possibly, social reasons). Second-best efficient freedom egalitarianism should then be defined. Among them are maximins and leximins in (potential) freedom. In the circumstances just noted, efficient (and core) allocations are freedom-ordered. Hence, there are least free and strictly least free individuals, second least free individuals when these are removed, and so on. This provides the basis for maximins and leximins.

If least free individuals[12] are unique for each efficient allocation, there are four basic related concepts of efficient maximin. An efficient maximin, indeed, can be an efficient allocation whose least free individual is related to that of each other efficient allocation by one of the four relations: she is freer or no less free than the other, or the other is less free or no freer than her (no freer means that either the individual is no less free or the other is less free). The freer maximin is unique if it exists. There cannot be both a no less free and a less free maximin. Least free individuals for allocations which are not a less free or a freer maximin are freer with and prefer this maximin. The least free individuals of no less free maximins are equally free.

If least free individuals are not unique for some allocations, more alternative concepts are possible. They rest on the fact that the least free individuals with each allocation are equally free, and freedom is comparable among groups of equally free individuals. Among the possible concepts, the uniform maximins where all the least free individuals with each allocation are treated alike have properties analogous to those the case with single least free individuals.

1.2.6. Realizations.

Finally, a main phenomenon consists of the interferences between the structure and the moral status of constraints and possibilities, and this is in particular crucial for the realization of the optimum or just solutions. Efficient "realistic" equal freedom is guaranteed

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[12] Or strictly least free, or minimally least free.

by individuals' independent choices in domains to which they are entitled. A notable application is classical liberalism in which the individuals are entitled to their own capacities. Symmetrical possibilities (i.e., allowing permutations of individuals' allocations), in addition to self-entitled personal possibilities, entail that allocations in the core are freedom-ordered. This symmetry can also be a required rule since it amounts to equal interferring freedoms.

These concepts, properties, and results, and other related ones, will be presented and discussed. Section 1.3 will define the basic technical issues and concepts. Section 2 defines equal-freedom allocations and the freedom comparisons, it shows the basic properties of these concepts, and the characteristic numbers and situations derived from the pairwise comparisons. Freedom-ordered allocations are analyzed in section 3 which shows their basic properties, those of least free and freest individuals, the layer structures of these allocations, the questions of the existence of "less free" cycles and of "improving permutations", and the relations with efficient and core allocations. The relations between potential freedom comparisons and preferences are shown in section 4. Section 5 then shows the various concepts of freedom maximins and leximins. Section 6 considers the consequences of the structure of the constraints and possibilities, and of their moral status of accountability and entitlement. The longest proofs are gathered in section 7.

1.3 Basic concepts and first notations

Let N denote the set of individuals in number $|N| \geq 2$, $i, j, \ldots$,N denote individuals, and $I, J, \ldots \subseteq N$ denote populations or groups. $J \subset I$ is a subgroup of I. The number of individuals in I is $|I|$, and $|I| < \infty$ means that this number is finite (this distinction will turn out to be crucial).

Let A denote the set of individual allocations with $|A| \geq 2$, $x_i 0 A$ denote an individual allocation for individual $i 0 N$, and $x_I = \{x_i\}_{i0\mathrm{I}}\ 0\ A^{|I|}$ denote an allocation for population (group) $I \subseteq N$. The issue of whether a given individual can or cannot have certain allocations will be specifically discussed later. Given x_i and $J \subset I$, $x_J = \{x_i\}_{i0\mathrm{J}}$, the projection of x_I on J (or the restriction of x_J to J), is a "suballocation" of x_I .

A complete strict ordering (or strict ranking) of a set of individuals $I \subseteq N$ with the binary relation will be denoted as (,I). The properties of this relation are the classical nonreflexivity, nonsymmetry, transitivity, and completeness .

$X_i \subseteq A$ will denote a domain of choice of individual i 0 N, and $X_I = \{X_i\}_{i0\mathrm{I}}$ is a *profile* of (independent) domains of choice for the population $I \subseteq N$. Individuals i and j are equally free if $X_i = X_j$, and individual i is no less free than individual j if $X_i \supseteq X_j$. The profile of domains X_I for $I \subseteq N$ is *equal* if all the X_I for i 0I are identical. It is *embedded* if there exists a complete strict ordering of the set I, (,I), such that i, j 0I and i j implies $X_i \supseteq X_j$. An equal profile is a particular embedded profile (the ordering can be anything). Denote as EQ and EM the sets of equal and embedded domain profiles, respectively, for any $I \subseteq N$. Then,

X_I 0 $EQ \Leftrightarrow (i0\mathrm{I} \Rightarrow X_i = Y \subseteq A)$,

X_I 0 $EM \Leftrightarrow [\exists(\ ,\mathrm{I}) : i,j\ 0\ I$ and $i\ \ j \Rightarrow X_i \supseteq X_j]$,

$EQ \subset EM$.

Clearly, if $J \subset I$ and X_J denotes the restriction of X_I to J (the projection of X_I on J), X_I 0 $EQ \Rightarrow X_j$ 0 EQ and X_I 0 $EM \Rightarrow X_J$ 0 EM.

If individual i0N is given a domain of choice $X_i \subseteq A$, she chooses as her individual allocation an element of her choice set $c_i(X_i)$, $x_i 0 c_i(X_i) \subseteq X_i$. Since the allocations will be directly evaluated in comparing domains of choice, but the (indirect) justice of the allocations will be considered (they are the situational variables), the basic tool will be the converse correspondence: The *freedom set* of individual i with allocation x_i is the set of possible domains of choice for i (subsets of A) from which the individual can choose x_i. This is $F_i(x_i)$ defined as:

$$F_i(x_i) \stackrel{d}{=} \{X_i \subseteq \mathrm{A}: x_i 0 c_i(X_i)\},$$

or, given that x_i0A and $X_i \subseteq \mathrm{A}$,

$$X_i 0 F_i(x_i) \stackrel{d}{\Leftrightarrow} x_i 0 c_i(X_i).$$

( ) denoting the set of parts of a set, c_i is a function $(A) \rightarrow (A)$ and F_i is a function $A \rightarrow [\ (A)]$.

For y 0A and denoting as $\{y\}$ the singleton set, $\{y\} = c_i(\{y\})$ and $\{y\}$0 $F_i(y)$.

For any $I \subseteq N$, x_I 0$A^{|I|}$, or X_I0[$(A)]^{|I|}$, denote as $c_I(X_I)=\{c_i(X_i)\}_{i0I}$ and $F_I(x_I) = \{F_i(x_i)\}_{i0I}$. Then, allocation x_I is a possible choice for domain profile X_I , or X_I is a possible domain profile for allocation choice x_I , or allocation x_I and domain profile X_I are congruent to each other, when the equivalent relations hold:

$x_I \in c_I(X_I) \Leftrightarrow X_I \in F_I(x_I)$.

We will later define equal-freedom and freedom-ordered allocations as allocations which can respectively be chosen from equal and embedded domain profiles.

Define also as

$$E_I(x_I) \overset{d}{=} 1_{i\in I} F_i(x_i) \qquad (1)$$

the *equal-freedom set* of domains for allocation x_I : the possible equal domain profiles for allocation x_I have each identical domains $X_i \in E_I(x_I)$ for all $i \in I$, and allocation x_I is equal-freedom if $E_I(x_I) \neq \emptyset$, that is, $F_I(x_I)1EQ \neq \emptyset$. Of course, for $J \subset I$ and the suballocation x_J of x_I, $E_J(x_J) \supseteq E_I(x_I)$.

These concepts, derived from the choice sets $c_i(X_i)$, suffice for all the following concepts and properties. However, it is possible to relate the following concepts and properties to the classical concept of individual preferences. Then, individual $i \in N$ will be endowed with the preordering R_i of A, P_i and I_i will respectively denote the corresponding strict preference and indifference (the antisymmetrical and the symmetrical parts of R_i), and the choice set $c_i(X_i)$ will be the set of maximal elements of R_i on X_i :

$x_i \in c_i(X_i) \Leftrightarrow x_i \in X_i$ and ($x_i' \in X_i \Rightarrow x_i R_i x_i'$).

Moreover, we will also introduce the possibility of specific limitations on the individuals' possible allocations (in addition to the X_i). They will be denoted as D_i, for each individual i. $D_i \subseteq A$. If all D_i are identical, the set A can just be taken as the same set. Otherwise, the two alternatives previously discussed can exist. They will lead to the following modifications. If the differences in D_i are considered as a priori unjustified from the point of view of justice, the X_i considered will have to be restricted to $X_i \subseteq D_i$. If, by contrast, the individuals are entitled/accountable for their own D_i, the choice sets will have to be $c_i(X_i \; 1 \; D_i)$, and the rest of the analysis will remain unchanged. The case with no restrictions or identical D_i is a particular case of the other two. But the case with self entitlement/accountability can also be reduced to this case in replacing the preference orderings R_i by new orderings $R_i^{'}$ such that $>,>^{'},D_i \Rightarrow (>R_i>^{'} \Leftrightarrow > R_i^{'} >^{'})$ and $>,D_i, >^{'} \notin D_i \Rightarrow > P_i^{'} >^{'}$ (the “sour grapes” transformation). This case is called the“realistic” case. The basic concepts for the case with no different D_i extend to both other cases. But, moreover, all properties with no different (and explicit) D_i will also hold for the “realistic” case. The question of different individual possibilities D_i will be particularly analyzed in section 6.

The existence of other constraints on the allocations x_I will also be considered, with particular interest in the properties of symmetry of these possibilities, that is, of their allowing permutations of individual allocations among the individuals, and in the efficient and core allocations (subsets E and $C \subseteq E$ respectively).

2. EQUAL FREEDOM AND FREEDOM COMPARISONS.

2.1 Equal-freedom allocations

Definition

Allocation x_I for group I is an *equal-freedom allocation* when it can result from individuals’ free choices in identical domains of choice; that is, denoting as EF the set of equal-freedom allocations (for whatever I),

$$x_I \in EF \overset{d}{\Leftrightarrow} E_I(x_I) \neq \emptyset \Leftrightarrow F_I(x_I) \; 1 \; EQ \neq \emptyset. \qquad (2)$$

Clearly, $x_I \in EF$ and $J \subset I$ implies $x_j \in EF$, since $J \subset I \Rightarrow E_J(x_J) \supseteq E_I(x_I)$. But the more general converse will be shown:

Proposition 1

An allocation for a population is equal-freedom if and only if it is equal-freedom for all subgroups, and if and only if it is equal-freedom for all pairs, of individuals of this population. That is,

$x_I \in EF \Leftrightarrow (J \subset I \Rightarrow x_J \in EF)$,

and

$x_I \in EF \Leftrightarrow (J \subset I$ and $| J | = 2 \Rightarrow x_J \in EF)$

or

$x_I \in EF \Leftrightarrow [i, j \in I \Rightarrow (x_i, x_j) \in EF]$.

The sufficiency of equal-freedom for each pair (and for each strict sub-group) for equal-freedom for the group are not a priori obvious.

2.2 Potential freedom comparisons

Pairwise freedom comparisons "as free as" and "no less free than"

Let $X_k \subseteq A$ denote a domain of choice for individual k. If, for two individuals i and j, $X_i = X_j$, individuals i and j are equally free. And if $X_i \supseteq X_j$, individual i is no less free than individual j.

Definition: potentially as free and no less free.

But the situational variables are the individual allocations x_i. And individual i is indifferent among being attributed x_i or any domain of choice $X_i \in F_i(x_i)$, from the assumptions. Hence, a relevant concept is that of potential freedom comparisons:

- Individual i with x_i is potentially *as free as* individual j with x_j when there exist $X_i \in F_i(x_i)$ and $X_j \in F_j(x_j)$ such that $X_i = X_j$, that is, $F_i(x_i)1F_j(x_j) \neq Ø$.
- Individual i with x_i is potentially *no less free than* individual j with x_j when there exist $X_i \in F_i(x_i)$ and $X_j \in F_j(x_j)$ such that $X_i \supseteq X_j$.

This will be written as, respectively,

$$x_i \; AF \; x_j \overset{d}{\Leftrightarrow} \exists \; X_i \in F_i(x_i), X_j \in F_j(x_j) : X_i = X_j, \qquad (3)$$

$$x_i \; NLF \; x_j \overset{d}{\Leftrightarrow} \exists \; X_i \in F_i(x_i), X_j \in F_j(x_j) : X_i \supseteq X_j. \qquad (4)$$

Properties.

These binary relations clearly have the following properties.

The relation AF is reflexive ($x_i \; AF \; x_i$) and symmetrical:

$x_i\ AF\ x_j \Leftrightarrow x_j\ AF\ x_i \Leftrightarrow (x_i, x_j) \in EF$.

From proposition 1, $x_I \in EF \Leftrightarrow (i,j \in I \Rightarrow x_i\ AF\ x_j)$.

The relation *NLF* is reflexive ($x_i\ NLF\ x_i$).

$x_i = x_j$ implies $x_i\ AF\ x_j$, $x_i\ NLF\ x_j$ and $x_j\ NLF\ x_i$, since $\{x_i\} \in F_i(x_i)$, $\{x_j\} \in F_j(x_j)$, and hence $\{x_i\} = \{x_j\} \in F_i(x_i) 1 F_j(x_j)$.

Finally, $x_i\ AF\ x_j \Rightarrow x_i\ NLF\ x_j$ and $x_j\ NLF\ x_i$. But the converse will also be shown, and so:

Proposition 2:

Two individuals with given allocations are as potentially free as each other if and only if each is potentially no less free than the other. That is,

$x_i\ AF\ x_j \Leftrightarrow x_i\ NLF\ x_j$ and $x_j\ NLF\ x_i$.

The converse relation is not a priori obvious.

There results: $x_I \in EF \Leftrightarrow (i,j \in I \Rightarrow x_i\ NLF\ x_j)$.

Definition: potentially less free and freer

The definition of *NLF* entails several further binary relations.

Given two individuals *i* and *j* with respective allocations x_i and x_j, and for potential freedom comparisons,

- Individual *i* is *less free* than individual *j* ($x_i\ LF\ x_j$) when she is not no less free than her,
- Individual *i* and individual *j* are not equally free when each is not as free as the other ($x_i\ NAF\ x_j \Leftrightarrow x_j\ NAF\ x_i$),
- Individual *i* is *freer* than individual *j* ($x_i F x_j$) when she is no less free than her and they are not equally free, or, equivalently, *i* is no less free than *j* and *j* is less free than *i*.

That is,

$$x_i\ LF\ x_j \overset{d}{\Leftrightarrow} \text{no } (x_i\ NLF\ x_j),$$

$$x_i\ NAF\ x_j \Leftrightarrow x_j\ NAF\ x_i \overset{d}{\Leftrightarrow} \text{no } (x_i\ AF\ x_j) \Leftrightarrow \text{no } (x_j\ AF\ x_i) \Leftrightarrow x_i\ LF\ x_j \text{ or } x_j\ LF\ x_i, \qquad (6)$$

$$x_i\ F\ x_j \overset{d}{\Leftrightarrow} x_i\ NLF\ x_j \text{ and } x_i\ NAF\ x_j \Leftrightarrow x_i\ NLF\ x_j \text{ and } x_j\ LF\ x_i, \qquad (7)$$

in using proposition 2.

Then, $x_i\ NLF\ x_j \Leftrightarrow x_i\ AF\ x_j$ or $x_i\ F\ x_j$.

Moreover, individual *i* can also be *no freer* than individual *j*:

$$x_i\ NF\ x_j \overset{d}{\Leftrightarrow} \text{no } x_i\ F\ x_j\ , \tag{8}$$

which implies

$x_i\ NF\ x_j \Leftrightarrow x_i\ LF\ x_j$ or $x_j\ NLF\ x_i \Leftrightarrow x_i\ AF\ x_j$ or $x_i\ LF\ x_j$.

The last possible situation in a pair[13] is that where the individuals are mutually less free, $x_i\ LF\ x_j$ and $x_j\ LF\ x_i$, with its contrary $x_i\ NLF\ x_j$ or $x_j\ NLF\ x_i$.

More generaly, a “less free allocation” is an allocation with which each individual is less free than each other: $i,j\in I \Rightarrow x_i LF x_j$. Allocations not having this property will eventually be interesting.

One easily sees that the binary relations *LF* is antireflexive, the binary relation *F* is antireflexive and antisymmetrical, and the binary relation *NF* is reflexive. All binary freedom relations have a priori no property of transitivity[14].

2.4 Characteristic numbers and the freedom comparison of allocations.

These sets of binary relations lead to a number of characteristic numbers and individuals' situations which provide criteria for the selection of allocations. Consider a given population with *n* individuals *i*, *j*, etc., and an allocation $x = \{x_i\}$ to this population. The basic numbers are numbers of binary relations of a certain type either to, from, or with an individual, or globally in the population. We will then denote, for a binary relation ϕ, as

$n_i^+(\phi)$ = number of j (or of $j \neq i$) such that $x_i\ \phi\ x_j$,

$n_i^-(\phi)$ = number of j (or of $j \neq i$) such that $x_j\ \phi\ x_i$,

$n_i(\phi) = n_i^+(\phi) - n_i^-(\phi)$,

$N(\phi) = \Sigma\, n_i^+(\phi) = \Sigma\, n_i^-(\phi)$ = number of relations ϕ.

We have $\Sigma\, n_i(\phi) = 0$.

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[13] That is, without considering substitutions such as the situation of individual *i* if she had allocation x_j, as it will be done shortly.

[14] The relations with strict inclusion of the domains of choice will not be used.

If relation ϕ is symmetrical, $n_i^+(\phi) = n_i^-(\phi)$ and $n_i(\phi) = 0$: this is the case for $\phi = AF$.

In denoting non-ϕ as $N\phi$,

$n_i^+(\phi) + n_i^+(N\phi) = n$ (or n-1),

$n_i^-(\phi) + n_i^-(N\phi) = n$ (or n-1),

$n_i(N\phi) = - n_i(\phi)$

$N(N\phi) = n^2 - N(\phi)$ (or $n \cdot (n-1) - N(\phi)$).

For two relations ϕ, ϕ_1 and ϕ_2, and $\nu = n_i^+$, n_i^-, or N, $\nu(\phi_1)$ and $\nu(\phi_2)$ are not smaller than ν (ϕ_1 and ϕ_2) and not larger than ν (ϕ_1 or ϕ_2).

The numbers $n_i^+(\phi)$ and $n_i^-(\phi)$ can run from 0 to n, and from 0 to n-1 if ϕ is nonreflexive or if the j considered in the definition exclude i. Correspondingly, $n_i(\phi)$ can run from n (or n-1) to $-n$ (or $-n$+1), and $N(\phi)$ can run from 0 to n^2 or $n \cdot (n-1)$. If ϕ is symmetrical (such as AF or NAF) the number of relations ϕ without repetition or reflexion is $N'(\phi)$ with $0 \leq N'(\phi) \leq C_n^2$, and $N'(\phi) + N'(N\phi) = C_n^2$. One has $2N'(AF) \leq N(NLF)$.

$x \in EF$ is equivalent to $n_i^+(LF) = n_i^-(LF) = 0$ for all i, $N(LF)=0$, $N'(NAF)=0$, $N'(AF)=C_n^2$, and other derived values. Hence, the numbers $N(LF)$, $N'(NAF)$, $N(LF)/n \cdot (n-1)$, $N'(NAF)/C_2^n$, or $\sum |n_i(LF)|$, which are non-negative, can be taken as "distances" of the allocation to equal-freedom, or *indexes of freedom inequality*. Among them $N(LF)/n \cdot (n-1)$ and $N'(NAF)/C_2^n$ are between 0 and 1, which they can reach, and they can be taken as indexes of *relative freedom inequality*. The number $N(LF)$ and $N(LF)/n \cdot (n-1)$ are particularly worthy for this purpose [15]. Similarly, *degrees of freedom equality* can be measured as $N(NLF)$ or $N'(AF)$, or as the numbers between 0 and 1 $N(NLF)/n^2$ or $N(NLF)/n \cdot (n-1)$ (according as the retained definition of $N(NLF)$), or $N'(AF)/C_2^n$. The ideal of equal freedom leads to the selection of allocations x which minimize these distances or inequalities, or maximize these degrees of equality, in sets of allocations x which can be pairs, possible x, efficient x, the core (see below), etc.

[15] A number similar to $N(LF)$ is used as an "envy index" by Feldman and Kirman (1974).

Less free allocations are characterized by $N(NLF) = 0$ and $N(LF) = n\cdot(n-1)$, and they satisfy $N^{'}(AF) = 0$.

Consider individuals' situations with *LF* or *NLF* as ϕ. Then, $n_i^+(LF)$ is the number of individuals *j* such that $x_i LF x_j$, and $n_i^-(LF)$ is the number of individuals *j* such that $x_j LF x_i$. Individuals *i* such that $n_i^+(LF) = 0$ are those who are less free than no other (no less free than all others): they are called the *freest* individuals. Individuals *i* such that $n_i^-(LF) = 0$ are those such that no other is less free than them (all others are no less free then them): they are called the *least free* individuals. Such individuals may exist or not (see next section). Among the freest, those with the largest $n_i^-(LF)$ (number of people less free than them) are the *maximally freest*. Among the least free, those with the largest $n_i^+(LF)$ (number of people to whom they are less free) are the *minimally least free*. A freest individual *i* with $n_i^-(LF) > 0$ is a *strictly freest* individual (at least one other is less free than her, and the individual is freer than these persons). A least free individual *i* with $n_i^+(LF) > 0$ is a *strictly least free* individual (she is less free than at least one other, and these others are freer than her). An individual can be both a freest and a least free, but this cannot occur if she is strictly freest or strictly least free.

The *globally least free* and *globally freest* are the individuals with highest and lowest $n_i(LF)$, respectively. They are not a priori least free or freest, but if they are, they more specifically are minimally less free and maximally freest, respectively.

The "less and most as free" individuals are the individuals with lowest and highest $n_i^+(AF) = n_i^-(AF)$, respectively.

For any allocation *x*, there always exist globally least free and globally freest individuals, and less as free and most as free individuals.

For $x \in EF$, all individuals are at once freest, least free, maximally freest, minimally least free, globally freest and least free, and less and most as free. Hence, in particular, the distinctions of least free, minimally least free, and globally least free can be used to define principles of maximin and leximin: one can choose allocations x which provide a better situation as regards the freedom of these individuals, or which minimize their number, in comparisons between alternative allocations. The following sections will provide a number of examples of such principles[16].

Moreover, all the definitions of comparisons of individual freedoms can apply to the comparison of the freedoms of the same individual in different situations, in the present case of her potential freedoms with different individual allocations, as binary relations between, say, x_i and $x_i^{'}$. We will see that $x_i F x_i^{'} \Leftrightarrow x_i^{'} LF x_i$ (section 4.3).

Then, allocation x is said to *freedom-dominate* allocation $x^{'} = \{x_i^{'}\}$ when $x_i\ NLF\ x_i^{'}$ for all i and $x_i F x_i^{'}$ for at least one i. Clearly, this binary relation between allocations is antireflexive and antisymmetrical (as the relation F is). A possible allocation which is not so dominated by other possible allocations is a priori to be sought (this will shortly be justified).

Another comparison is majority. The same result is obtained whether the binary relation used is NLF or F. For a binary relation ϕ and two allocations x and $x^{'}$ to the same population, denote as $N(x, x^{'}, \phi)$ the number of i such that $x_i \phi x_i^{'}$. Allocation x wins by majority over allocation $x^{'}$ for relation ϕ when $N(x, x^{'}, \phi) > N(x^{'}, x, \phi)$, or $n(x, x^{'}, \phi) \stackrel{d}{=} N(x, x^{'}, \phi) - N(x^{'}, x, \phi) > 0$. But $n(x, x^{'}, NLF) \equiv n(x, x^{'}, F)$ as one easily sees. This binary relation is antireflexive and antisymmetrical.

3. FREEDOM-ORDERED ALLOCATIONS

The following concepts, structures, and properties will be important, notably for defining the various concepts of maximin and leximin in freedom.

3.1 Embedded domains

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[16] Other analyses also use the "median" (in a particular sense) individuals for each relations, that is, for relation ϕ, the individuals i such that the absolute value $|n_i(\phi)|$ is the lowest, for $\phi = LF$ (or NLF), or F.

Definition: freedom-ordered allocations.

An allocation is *freedom-ordered* when the individuals can be ranked in such a way that each is no less free than each individual of lower rank.

That is, *FO* denoting the set of freedom-ordered allocations (for any population *I*),

$$x_I \in FO \overset{d}{\Leftrightarrow} \{\exists\ (\prec, I): i,j \in I \text{ and } i \prec j \Rightarrow [x_i\ NLF\ x_j \Leftrightarrow \exists\ X_i \in F_i(x_i), X_j \in F_j(x_j) : X_i \supseteq X_j]\}.$$

Clearly, $EF \subset FO$.

Clearly also, a suballocation of a freedom-ordered allocation is freedom-ordered:

$x_I \in FO$ and $J \subset I \Rightarrow x_J \in FO$.

The following property will be shown:

Proposition 3

An allocation is freedom-ordered if and only if it can result from individuals' choices in an embedded profile of individual domains of choice.

That is,

$$x_I \in FO \Leftrightarrow F_I(x_I)\ 1\ EM \neq \phi \Leftrightarrow \exists\ X_I \in F_I(x_I) \text{ and } (\prec, I): (i,j \in I \text{ and } i \prec j \Rightarrow X_i \supseteq X_j).$$

The difference with the definition is that the same domain of choice is used for each individual in each pairwise comparison, while this is not a priori the case in the definition. Proposition 3 says that if x_I is freedom-ordered, the same domain X_i can be taken for defining the relation *NLF* for each pair including *i*. The existence of such a freedom-ordered profile of embedded possible domains obviously implies the definition, but the converse is not obvious.

Proposition 3 and the definition of a freedom-ordered allocation respectively correspond to the definition of an equal-freedom allocation and proposition 1.

Clearly also, if $J \subset I$ and $x_I \in FO$, the projection on *J* (or restriction to *J*) of an embedded profile of proposition 3, X_I, for the population *I*, also is such an embedded profile X_J for the subpopulation *J* with the suballocation x_J.

If the number of individuals is finite and is n, the number of strict orderings of the individuals which can define a freedom-ordered allocation, ν, varies from 0 to $n!$ The case $\nu=0$ means that the allocation is not freedom-ordered (for instance if the individuals of one pair are each less free than the other – a necessary and sufficient condition will shortly be shown). The case $\nu=n!$ corresponds to equal-freedom allocations. The number ν for a given allocation can be taken as a degree of freedom ordering of this allocation.

Moreover, for $x_I \in FO$, and with $n=|I|$, $[n\cdot(n-1)]/2 \leq N(NLF) \leq n\cdot(n-1)$ (excluding relations x_i NLF x_i) and $0 \leq N(LF) \leq [n\cdot(n-1)]/2$.

3.2 Least free and freest

Definition

An individual is

- *least free* if no other is less free than her,
- *freest* if she is less free than no other,
- *strictly least free* if she is least free and no freest (hence one other is freer than her),
- *strictly freest* if she is freest and no least free (hence she is freer than one other).

Clearly,

- Least free individuals are equally free,
- Freest individuals are equally free,
- An individual freer than a strictly least free is not least free,
- A strictly freest individual is freer than non-freest individuals only,
- An individual both least free and freest is as free as each individual – all individuals are in this case if and only if the allocation is equal-freedom.

The following property of existence will be shown:

Proposition 4

With a freedom-ordered allocation to a finite number of individuals, there exist least free individuals and freest individuals, and, if the allocation is not equal-freedom, there exist strictly least free individuals and strictly freest individuals. The latter are respectively the *least free and the freest individuals for the suballocation to the set of individuals minus the individuals who are both least free and freest.*

Proposition 4 applies to all suballocations of the considered allocation, and to all suballocations to a finite number of individuals of any freedom-ordered allocation.

More generally, each least free and freest individual can be respectively characterized by the number of (non-least free) individuals who are freer than her and of (non-freest) individuals to which they are freer. This establishes a hierarchy among least free and among freest. The extremes are the *minimally least free* who are the least free individuals with the largest number of (non-least free) individuals who are freer than them, and the *maximally freest* who are the freest individuals with the largest number of (nonfreest) individuals to which they are freer (the other two extremes of these numbers are less interesting).

3.3 The layer structure

The receivers of a freedom-ordered allocation can be partitioned into ranked layer subsets of equally free individuals who are no less free than individuals of lower layers and such that individuals of higher layers one no less free than them. The individuals of the same layer can have identical domains in the profiles of embedded domains of individual choice. There may be a number of possible such arrangements. Two are particularly important. One is constituted in considering the least free individuals, then the least free of the remaining set, and so on. The other is constituted in considering the freest individuals, then the freest of the remaining set, and so on.

In the first, "least free", structure, each individual is freer than at least one individual of the immediately lower layer (except for the least free individuals). Indeed, is this were not the case she would be in this lower layer. Specifically, when this lower layer is constituted in becoming the set of least free individuals, since the considered individual is not included in it, at least one individual not yet discarded is less free than her. And since she becomes least free when this lower layer is also discarded, these individuals belong to this lower layer. Then, since she also is no less free than these individuals, she is freer than them. The highest layer is constituted of the strictly freest individuals who are freer than at least one individual of the immediately lower layer.

In the second, "freest", structure, for each individual there is at least one individual of the immediately higher layer who is freer than her (except for the freest individuals). Indeed, if this were not the case this individual would be in this higher layer. The reasoning parallels that of the previous case. The lowest layer is constituted of the strictly least free individuals for which at least one of the individual of the second lowest layer is freer.

3.4 Less free cycles

Definition

For an allocation, a *less free cycle* is a closed sequence (cycle) of individuals such that each is less free than the next. That is, for x_I it is a set $i,j,k,\ldots,\mathrm{P} \in I$ such that $x_i\ LF\ x_j$, $x_j\ LF\ x_k$,…, $x_{\mathrm{P}}\ LF\ x_i$. The following property will be shown:

Proposition 5

1) *There is no less free cycle with a freedom-ordered allocation.*
2) *If there is no less free cycle and the number of individuals is finite, the allocation is freedom-ordered.*

Hence, with a finite number of individuals, the properties of freedom-order and of the absence of less free cycles are equivalent.

3.5 Improving permutations and efficiency.

A *freedom-improving permutation* is a permutation of their individual allocations among the individuals of population I such that, if $\pi(i)\in I$ denotes the individual whose allocation goes to individual $i\in I$, then $x_i LF x_{\pi(i)}$ if $\pi(i) \neq i$; that is, each individual whose allocation changes becomes potentially freer ($x_i LF x_i' \Leftrightarrow x_i' F x_i$ for comparisons of the same individual's potential freedom with two different allocations, as it will shortly be shown).

For a given allocation x_I and a freedom-improving permutation π, the i such that $\pi(i)\neq i$ constitute a less free cycle if $|I|<\infty$. Hence, since $x_I\in FO$ implies that there is no less free cycle, $x_I\in FO$ and $|I|<\infty$ imply that there is no freedom-improving permutation. Conversely, if $|I|<\infty$, the absence of possible freedom-improving permutations implies that of less free cycles, and hence implies that x_I is freedom-ordered; however, the consequences of this latter relation depends on the actual possibilities of permutations. We will thus make this issue explicit, in anticipating here on the issues which will be fully discussed in section 6.

The relation "less free" now considered will be, if the individuals have different domains of possible allocations, $x_i\in D_i$ for individual i, the case where each is entitled to her possibilities and hence is accountable for their limitation. Then, it will be shown in section 6 that for this LF, denoted as LF^+ and qualified as "realistic",

$x_i\ NLF^+x_j \Leftrightarrow x_iR_ix_j$ or $x_j\notin D_i$,

$x_i\ LF^+x_j \Leftrightarrow x_jP_ix_i$ and $x_j\in D_i$,

and, for $x_i\in D_i$,

$x_i\ NLF^+\ x_i' \ \Leftrightarrow\ x_iR_i\, x_i'$ or $x_i'\notin D_i$,

$x_i\ LF^+\ x_i' \ \Leftrightarrow\ x_i' F^+ x_i \ \Leftrightarrow\ x_i' P_i x_i$ and $x_i' \in D_i$.

The case with no explicit D_i and the case where all D_i are identical amount to the same (one can take $A = D_i$). They are a particular case of the previous case. But, conversely, the previous case can be reduced to that with identical D_i in replacing the orderings R_i with the derived "sour grapes" orderings R_i'.

Denote as $D = \{x_I\in A^{|I|}\ :\ (x_I\in D \Leftrightarrow x_i\in D_i,\ \forall i\in I)\}$. There may be other constraints on x_I, denoted as $x_I\in P$. The total limitation on x_I is $x_I \in D1P$.

If $Z \in (A^{|I|})$ is a set of possible x_I, denote as $E(Z) \subseteq Z$ the set of corresponding Pareto-efficient x_I.

Then, $x_I \in E(D1P) \Leftrightarrow x_I \in D1P$ and for no $x_I' \in D1P$, $x_i' R_i x_i$, $\forall\ i\in I$, and $x_i' P_i x_i$ for at least one $i\in I$. But, since $x_i, x_i' \in D_i$, $x_i' R_i x_i \Leftrightarrow x_i' NLF^+ x_i$, and $x_i' P_i x_i \Leftrightarrow x_i\ LF^+ x_i' \Leftrightarrow x_i' F^+ x_i$.

Therefore, (Pareto-)efficiency is identical when expressed with freedom comparisons or with preferences as they usually are. We will just say efficient and efficiency.

Assume domain P to be symmetrical, that is, $x_I \in P \Leftrightarrow \{x_{\pi(i)}\} \in P$ for all permutations π of the set I. When considering permutations, the only possible remaining limitations can solely come from domain D.

Hence, if there is a freedom-improving permutation π, the relations

$x_i \, LF^+ x_{\pi(i)} \Leftrightarrow x_{\pi(i)} \, P_i \, x_i$ and $x_{\pi(i)} \in D_i$

for all $i \neq \pi(i)$ imply that

1) the permutation is actually possible since $x_{\pi(i)} \in D_i$,,
2) $x_{\pi(i)} \, P_i \, x_i$ for all these individuals i.

Therefore, if a freedom-improving permutation exists from allocation x_I,

1) this allocation is not efficient,
2) this allocation is not in the "core" in the sense that the group of individuals such that $\pi(i) \neq i$ can rearrange their allocations so as to be all both potentially freer and better off. Since the core is efficient, property 1 implies the intrinsically meaningful property 2.

Therefore, if allocation x_I is efficient, or is in the core, there is no freedom-improving permutation, and hence, if $|I|<\infty$, there is no less free cycle, and therefore $x_I \in FO$.

The following property thus holds:

Proposition 6 :

For a finite number of individuals, if nothing can prevent permutations except individual possibilities, and if individuals are entitled to their own possibilities or accountable for their limitations, or if they have identical domains of possibilities, then efficient and core allocations are freedom-ordered.

This is "realistic" freedom ranking, which amounts to the simple one if the individuals have identical domains. In brief, efficient and core allocations are realistically freedom-ordered if the individuals are accountable for the obstacles to permutations, and are finite in number.

3.6 Maximins

The best allocation certainly has to be efficient, for moral reasons, notably based on freedom, which will be recalled in section 4.1. Allocations may also have to be in the core, for two possible reasons, based on the fact that allocations not in the core can be destroyed by a unanimous "blocking coalition" rearanging its individual allocations. Indeed, if the individuals of a blocking coalition perform this rearangement, the allocation from which they do it is not stable, and the relevant concept of a possible allocation problably has to require that this allocation is stable. But there are two possible reasons for such actions of blocking coalitions not to be prevented. First, preventing this action may just not be possible. Second, preventing this action may be banned for a moral reason because subsets of individuals have the right to agree among themselves, exchange promises of action, and so transform their allocations. Moreover, this latter right may be embodied a priori in the selection of allocations, which thus a priori implements its possible effects, without the individuals actually considering its use (and the actual agreements may be costly, difficult, or impaired for any reason) – the ethics then is one of a "liberal social contract" implementation of these rights[17].

There may be no equal-freedom allocation which is possible, or efficient, or in the core. One then has to replace equal-freedom by a second-best freedom egalitarian principle respecting this constraint. The fact that efficient and core allocations are freedom-ordered provides opportunities in the family of maximins or leximins, which will be closely considered. We will then first focus on the least free, or strictly least free, or minimally least free individuals. However, there can be several such individuals. But then they are equally free, and this permits the comparison.

[17] See Kolm 1985, 1987a, 1987b, 1996a.

Indeed, if for two different populations $I,J \subseteq N$ allocations $x_I \in A^{|I|}$ and $y_J \in A^{|I|}$ are equal-freedom, one can define "as free as" and "no less free than" for these allocations as, respectively,

$$x_I \; AF \; y_J \overset{d}{\Leftrightarrow} E_I(x_I) \; 1 \; E_J(y_J) \neq \varnothing$$

and

$$x_I \; NLF \; y_J \overset{d}{\Leftrightarrow} \exists X \in E_I(x_I),\; Y \in E_J(y_J) : X \supseteq Y.$$

It will then be shown that:

Proposition 7

If $x_I, y_J \in EF$,

$x_I \; AF \; y_J \Leftrightarrow x_I \; NLF \; y_J$ and $y_J \; NLF \; x_I$.

Then, for a given population, denote, for each $x = \{x_i\} \in FO$, as M the set of least free (or strictly least free or minimally least free) individuals, and as x_M the corresponding suballocation of x. Another such allocation x' similarly has a M' and a $x'_{M'}$. Let E and C respectively denote the sets of efficient and core allocations, with $C \subseteq E \subseteq FO$. Then, G denoting E or C, a corresponding freedom maximin can be defined as $x \in G$ such that $x' \in G \Rightarrow x_M \; NLF \; x'_M$. All such x have equally free x_M. There are, however, a number of other related and relevant concepts of freedom maximins and leximins (including some which secure uniqueness of the solution). Section 5 will provide their definitions, comparisons, and properties.

4. RELATIONS BETWEEN POTENTIAL FREEDOM COMPARISONS AND PREFERENCES.

4.1 Relations

The foregoing comparisons of meaningful potential freedoms can be expresed in terms of the individuals' preferences, and of the individuals' possibilities when they differ. The cases with different individuals' possibilities will be explicitly considered in section 6.

The case where these possibilities are the same will be shown to lead to the following simple correspondences:

Proposition 8

8-1: $x_i\ LF\ x_j \Leftrightarrow x_j P_i x_i$,

8-2: $x_i\ NLF\ x_j \Leftrightarrow x_i R_i x_j$,

8-3: $x_i\ F\ x_j \Leftrightarrow x_i R_i x_j$ and $x_i P_j x_j$,

8-4: $x_i\ NF\ x_j \Leftrightarrow x_j P_i x_i$ or $x_j R_j x_i$,

8-5: $x_I \in EF \Leftrightarrow (i,j \in I \Rightarrow x_i R_i x_j)$,

8-6: $x_i\ AF\ x_j \Leftrightarrow x_i R_i x_j$ and $x_j R_j x_i$.

Proposition 8 proves propositions 1 and 2: 8-5 and 8-6 entail proposition 1, and 8-2 and 8-6 entail proposition 2.

4.2 Meanings of equal-freedom

Proposition 8-5 reveals that the equal-freedom principle with no limitations (or identical limitations) in individuals' possibilities amounts to none other than the classical criterion that no individual prefers any other's allocation to her own. In fact, this possible equality of liberty constitutes the basic and most important and meaningful reason for the ethical worth and for the importance of this property. Note that this criterion is often used without saying why it is important[18]. However, two types de justificatory notions seem to exist in the mind of people who consider it, and others can be suggested.

On the one hand, this criterion is often presented as a directly meaningful type of equity or fairness. But what is, then, the underlying rationale? Is there an ideal equalizand – the metaethical theory of justice suggests there should be one –, and, then, if this equality is not achieved, what is the reason for this ? Economists are prone to emphasize that solely

[18] This accounts for the variations in the names given to this criterion which was successively called "the exchange principle" by Tinbergen (1946), "equity" by Foley (1967), Kolm (1971) and others, "fairness" when associated with Pareto-efficiency by Varian (1974), "super-fairness" by Baumol (1986), and "envy-freeness" by many later authors. "Equity" can stand for *Equ*al *I*nstrumental *I*ndependent Liber*ty*.

ordinal preferences are used. This suggests that preferences, utility, satisfaction, etc… would be the ideal relevant end-value, but that difficulty in being more specific than ordinal and independent preferences constitutes the obstacle. Moreover, in this evaluation of allocations, the sole characteristic of individuals are their preference orderings. And if the individuals have identical preference orderings, then the application of the principle entails that the individual allocations are indifferent among themselves, with this ordering. But rationality implies equal treatment of equals[19], and hence the objective seems to be the level of preferences. But indifference with the common ordering does not mean same level of satisfaction (though the identity of preference orderings would be quite fortuitous if it were not derived from some identical satisfaction level function[20] – this satisfaction would then be interpersonally comparable, but it can remain ordinal, and hence be co-ordinal). In the end, such a conception probably sees personal satisfaction, "welfare", or happiness as the relevant item for the direct evaluation, with the obstacle being the notional non-comparability across individuals. But probably more can be introduced in the way of this comparability[21].

Another idea would be that the items directly relevant for justice are individuals' allocations, and the ideal is equal individual allocations. This would result from the irrelevance, for the considered direct evaluation, of individuals' eudemonistic capacities only, for a reason of individual entitlement to these capacities or accountability for their shortcomings, or of privacy of the feeling of satisfaction. But this ideal generally is not Pareto-efficient because of differences in individuals' preferences, and the considered criterion would be a second-best form which may permit efficiency. The criterion is indeed satisfied, in particular, by identical individuals' allocations. But the introduction of preferences should be justified (the various reasons for Pareto-efficiency will shortly be recalled). One may consider that equal individual allocations also constitute a particular case of identical domain of choice (the case where these domains vanish to a singleton), and hence the extension of equality in allocations which may permit Pareto-efficiency would be identical domains of choice, which leeds to the principle as shown above.

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[19] The fullest presentation of this point is in Kolm 1998, foreword, section 5.

[20] The exception to this remark would be that individuals solely care about one unidimensional item being "more" or "less" (possibly, but not necessarily, the quantity of a good), and the considered allocations x_i are the bundle of factors which determine this item.

[21] The fullest analysis of this suggestion is in Kolm 1998 (translation of 1971), foreword 1997.

But a common suggestion relates this principle to an absence of envy. Tinbergen (1946) discusses this aspect, and the logical relations with envy and jealousy are noted in Kolm (1971). More recently, this principle came to be often referred to as no-envy or envy-freeness, though users of this term usually do not state explicitly whether they actually refer to a sentiment of envy, or just to the formal structure of the criterion. The reference to actual envy has sometimes been suggested. And envy is both a standardly morally condemned sentiment and a painful one. The moral condemnation, however, would rather lead to discard this sentiment for normative considerations[22]. Yet, this painfulness and this moral condemnation hold for the most common type of envy, or strong envy, and this sentiment cannot be described by the preferences considered here, since an envious person is jointly influenced both by what she has and by what the people she envies have. And, indeed, a long line of studies have modelled envy as such a consumption externality[23]. Rarer and milder types of envy (like “I envy your youth”) would take us back to the previous conception of the criterion as a direct equity concept.

Hence, the essential value of this principle is its derivation from freedom justice. This leads to an apparent paradox: equity is solely expressed in terms of individuals’ preferences, while its essential value rests on the fact that preferences are discarded from relevant direct concern. This essential value is the reverse of the common conception which holds this criterion to be valuable because it solely is expressed in terms of individuals’ preferences, indeed of ordinal preferences without interpersonal comparaison. This principle is basically rational first-best eleutheristic (or freedom) justice, rather than second-best eudemonistic (or satisfaction) justice shunning interpersonal comparison of preferences – or second-best allocational justice trying to avoid inefficiency. Individual preferences then appear in the classical expression of the principle solely because the considered situational variables (the allocations) differ from the directly (or ultimately) justice-relevant variables (the freedoms), and individuals’ (potential) choices translate equal freedom in the space of allocations.

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[22] And to replace individual preferences in which this envy is correctly modelled by “laundered” preferences where the effect of this sentiment has been erased, which is technically possible (see Kolm 1991b, 1995).

[23] See the history in Kolm 1995.

4.3 Comparison of potential freedoms of the same individual.

In the potential freedom comparisons, the two individuals with their allocations can be the same individual with allocations which can be different. The relations then compare the potential freedoms of the same individual with two allocations. Formally, this amounts to the two individuals having the same preferences and personal possibilities, and presently solely the same preferences since personal possibilities are a priori assumed to be identical (the more general case will be considered in section 6). Hence, if x_i and $x_i^{'}$ are two individual allocations of individual i, substituting $x_i^{'}$ for x_j and R_i for R_j in proposition 8 provides the correspondence between potential freedom and preferences for the same individual, which turns out to simply be:

$x_i \, F \, x_i^{'} \Leftrightarrow x_i^{'} \, LF x_i \Leftrightarrow x_i P_i \, x_i^{'}$,

$x_i NLF \, x_i^{'} \Leftrightarrow x_i^{'} \, NF x_i \Leftrightarrow x_i R_i \, x_i^{'}$,

$x_i AF \, x_i^{'} \Leftrightarrow x_i I_i \, x_i^{'}$.

That is:

Proposition 9

For the same individual, potential freedom comparisons and preferences amount to the same.

Hence, each individual prefers to be freer, is indifferent between being as free, prefers or is indifferent to be no less free, and conversely, with this purely choice-instrumental and potential freedom. The same result will hold for the cases where individuals can have different possibilities if one assumes $x_i, x_i^{'} \in D_i$, since these differences do not intervene in these comparisons for the same individual.

4.4 Efficiency

One consequence is that the concepts of unanimous improvement and of Pareto efficiency are identical when expressed in terms of possible freedoms and in terms of preferences as they classically are. A possible allocation is Pareto efficient when an individual can become potentially freer solely if another becomes potentially less free.

Hence, Pareto efficiency is required for two reasons by a social ethic based on freedom. The first reason depends on the type of concepts used here: if a possible allocation is not Pareto efficient, all individuals can be made (potentially) freer, with the possible exception of some (but not all) who remain as free. The second reason is general. Indeed, if a possible allocation is not Pareto efficient, there exists another possible allocation that all individuals prefer to choose, with possible indifference for some individuals – but not for all. Hence, Pareto inefficiency constitutes an unnecessary constraint on society, whatever its reason. Thus, seeking higher (in inclusion sense) actual collective freedom requires Pareto efficiency [24]. Therefore, for all reasons we will henceforth require Pareto efficiency with priority. For short, we will just write efficiency and efficient.

It may be that no efficient allocation is equal-freedom. It may also be, more generally, that no possible allocation is equal-freedom, which implies that no efficient allocation is equal-freedom. This is a priori due to limitations in divisibility or in transferability of the items in individual allocations[25]. These properties are rather frequent occurrences. Their reasons may be physical: certain goods may be indivisible, or cutting them down may destroy them or make them useless or much less useful; divisible and transferable goods may be in short supply; personal capacities are not directly transferable. But these limitations may also have a social cause which has priority such as nondivision, inalienability, minimal or maximal allocation, due to moral, other norms, politics, or other power. Limitations of the set of individual allocations that an individual can have will later be explicitly considered (section 6), with different ethical treatments according as whether the individuals are, or are not, deemed accountable for their limitations and entitled to their possibilities. But we will begin with discarding this consideration, for reason of simplicity in presentation, because the concepts extend to the cases with explicit limitations, and because, for the case where the individuals are accountable for their limitations or entitled to their possibilities, all the properties of the simple presentations will have analogs and, indeed, the full structure can formally be reduced to the simple presentation.

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[24] If the inefficeincy is due to individuals' interaction, this justification of imposing efficiency relates to a notion of potential freedom or "liberal social contract".

[25] Efficient equal-freedom allocations exist with perfectly divisible and transferable goods and convex possibility sets and preferences (see Kolm 1971, 1995, 1996b).

With no efficient equal-freedom allocation, second-best eleutheristic justice should be defined, which will be an efficient second-best freedom egalitarianism. If, moreover, the allocation should be in the core because unanimous rearangements within coalitions, which would destroy the solution, cannot be prevented, or should not be prevented because they are a right (free exchange), or even should a priori be imagined for the same reason, then the search for an allocation is further restricted to the core, a subset of efficient allocations. Hence, the domain in which the solution is sought is either the set of efficient allocations E, or the core C. We will write it E for short (and because the core becomes the set of efficient and of possible allocations if the destructive – or potentially destructive – actions of coalitions are classified within the constraints of the problem). Since the social ethic positively values the considered freedom, as the individuals do, and since more or less free is defined both across individuals and for each individual, it is natural to seek solutions in the family of maximins. Efficient freedom maximins will thus now be defined. For short, the qualificative "potential" referring to freedoms will henceforth remain implicit. These definitions will rest on proposition 6 and on the structure of freedom-ordered allocations.

5. MAXIMINS

Practically, it is not uncommon that, in the actually possible allocations, the least free individuals are the same ones (no other individual would prefer to be in their place). If there is solely one such least free individual, then her preferences constitute a social ethical ordering, and the maximal elements of this preordering constitute the solution. However, more generally, the least free individuals (as defined in section 2.4) may not be the same in various efficient allocations, and there may be several least free individuals in a given allocation. In this latter case, the minimally least free individuals may be considered, and there may be solely one of them in each allocation, possibly the same one whose preferences can then become the social ethical ordering. We will successively consider the cases where each efficient allocation has solely one least free (or minimally least free) individual who, however, can vary from one allocation to the other, and the more general case where each efficient allocation can have several least free individuals. In all this section, *least free can alternatively be replaced by minimally least free*.

5.1 Minimal freedom comparisons with single least free.

Definitions

Consider a given population, not explicitely denoted, of individuals *i*, with allocations $x=\{x_i\}$. Least free (and minimally least free) individuals exist in freedom-ordered allocations, which are the case in the conditions stated in proposition 6. We now consider allocations with *single least free* (or minimally least free) individuals, denoted as *m*. From proposition 4, if *x* is freedom-ordered and not equal-freedom, individual *m* also is strictly least free, and hence it is not freest and $x_j \, F \, x_m$ for some $j \neq m$. Let *x* and x' denote two such allocations, with *m* and m' as respective such individuals. *Minimal comparisons* of *x* and x' are comparisons between x_m and $x'_{m'}$. The freedom comparaisons (or their expression in terms of preferences) are considered. Then, allocation *x* being *minimally as free as, no lest free than, less free than, freer than,* and *no freer than* allocation x' are respectively denoted and defined as :

$$xMAF\,x' \overset{d}{\Leftrightarrow} x_m AF\, x_{m'} \Leftrightarrow x_m R_m \, x'_{m'} \text{ and } x'_{m'} \; R'_{m'} \, x_m \,,$$

$$xMNLF\,x' \overset{d}{\Leftrightarrow} x_m NLF\, x'_{m'} \Leftrightarrow x_m \, R_m \, x'_{m'} \,,$$

$$xMLF\,x' \overset{d}{\Leftrightarrow} x_m LF\, x'_{m'} \Leftrightarrow x'_{m'} \, P_m \, x_m \,,$$

$$xMF\,x' \overset{d}{\Leftrightarrow} x_m F\, x'_{m'} \Leftrightarrow x_m \, R_m \, x'_{m'} \text{ and } x_m \, P_{m'} \; x'_{m'} \,,$$

$$xMF\,x' \overset{d}{\Leftrightarrow} x_m NF\, x'_{m'} \Leftrightarrow x'_{m'} \, P_m \, x_m \text{ or } x'_{m'} \; R'_{m'} \, x_m \,.$$

The relation *MAF* is symmetrical, and the relation *MF* is antisymmetrical.

From the definitions,

$$xMAF\,x' \Leftrightarrow xMNLF\,x' \text{ and } x'\,MNLFx \,,$$

$$xMF\,x' \Leftrightarrow xMNLF\,x' \text{ and } x'\,MLFx \,,$$

$$xMNLF\,x' \Leftrightarrow xMAF\,x' \text{ or } xMF\,x' \,,$$

$$xMNF\,x' \Leftrightarrow xMLF\,x' \text{ or } x'\,MNLFx \,,$$

$$xMNLF\,x' \Leftrightarrow \text{no } xMLF\,x' \,,$$

$$xMNF\,x' \Leftrightarrow \text{no } xMF\,x' \,.$$

If individuals m and m' are the same individual, the minimal ranking of x and x' is as this individual's preferences or potential freedom. But even when they are not the same individual, the relation *MF* will be shown to entail the following property:

Proposition 10

With unique least free individuals, such an individual becomes freer with a minimally freer allocation, and, more generally, with an allocation such that the present one is minimally less free than it.

That is, $x' MFx \Rightarrow xMLF\,x' \Rightarrow x'_m P_m x_m$.

5.2 Freedom maximin with single least free.

Definitions.

Consider a given population with allocations $x=\{x_i\}$. Assume the efficient allocations (set E) have single least free (or minimally least free) individuals denoted as m (and as m' for allocations x') – their reason for having least free individuals can be that they are freedom-ordered for the reasons noted earlier. There are four related concepts of *efficient freedom maximin* (*EFM*), respectively built up with the relations minimally (*M*) no less free (*NLF*), freer (*F*), less free (*LF*), and no freer (*NF*). These efficient maximin are $x \in E$ such that, for all other $x' \in E$, respectively:

1) For a *No less free efficient freedom maximin* (*NLFEFM*),

   $xMNLF\,x' \Leftrightarrow x_m R_m x'_{m'}$;

2) For a *Less free efficient freedom maximin* (*LFEFM*),

   $x' MLFx \Leftrightarrow x_m P_{m'} x'_{m'}$;

3) For a *Freer efficient freedom maximin* (*FEFM*),

   $xMF\,x' \Leftrightarrow xMNLF\,x'$ and $x' MLFx \Leftrightarrow x_m R_m x'_{m'}$ and $x_m P'_{m'}\ x'_{m'}$;

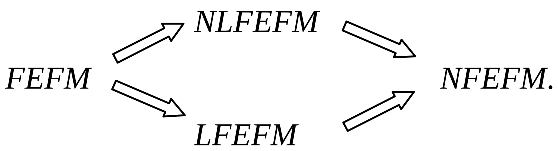
4) For a *No freer efficient freedom maximin* (*NFEFM*),

   $x' MNFx \Leftrightarrow xMNLF\,x'$ or $x' MLFx \Leftrightarrow x_m R_m x'_{m'}$ or $x_m P'_m\ x'_{m'}$.

The following properties directly result or will be shown:

Proposition 11

1) *A FEFM is unique.*
2) *There cannot be a NLFEFM and a LFEFM distinct.*
3) *The sets of FEFM and of NFEFM are respectively the intersection and the union of the sets of the NLFEFM and of the LFEFM. Hence, the implications of these properties are*



4) *The least free individuals of all NLFEFM are equally free. All NLFEFM are minimally as free as one another.*
5) *The least free individuals of the allocations which are not a certain LFEFM are freer with and prefer this LFEFM.*
6) *In particular, the least free individuals of the allocations which are not the FEFM if it exists, are freer with and prefer the FEFM.*
7) *The least free individuals of the allocations which are not NLFEFM are freer with and prefer certain other efficient allocations.*

The maximin choice should be the *FEFM* if it exists. It has, indeed, more dominating properties over other allocations than other *NFEFM*, and no such property less. If there is no *FEFM*, then a *NFEFM* is to be chosen. *NLFEFM* or *LFEFM* have a certain advantage of consistency over other *NFEFM :* the binary relations with the other allocations, which define them, are of the same type. Only one of these two categories can exist. *NLFEFM* allocations are all minimally equally free. The *LFEFM* allocations have the advantage that each makes freer, and is preferred by, the least free individuals in each other allocation. When no single allocation is selected in this way, a leximin can be applied in dropping the least free individuals and their allocations in the possibly selected allocations, and in comparing the second least free individuals, and so on.

Several least free individuals.

However, freedom-ordered allocations can in general have several least free individuals. Concepts of freedom maximin thus demand that the freedom of these least free groups be compared. These groups are constituted with individuals who generally differ across allocations, and whose number also generally differ. But these least free groups are each equal-freedom groups. This permits one to define the needed freedom comparisons.

5.3 Freedom comparisons of different equal-freedom groups.

Let $x_i, y_i \in A$ denote individual allocations for $i \in N$, $I, J \subseteq N$ denote two groups of individuals, $x_I = \{x_i\}_{i \in I} \in A^{|I|}$ and $y_J = \{y_i\}_{i \in J} \in A^{|J|}$ denote two allocations for these groups respectively, and assume each of these allocations to be equal-freedom: x_I, $y_J \in EF$, that is, $E_I(x_I) \neq \varnothing$ and $E_J(y_j) \neq \varnothing$. The comparison of (potential) freedoms of group I with allocation x_I and of group J with allocation x_J, initiated in section 3.6, will be completed with other relevant concepts. The following comparisons will be defined: equal freedom (as free as: *AF*), no less free than (*NLF*), and weakly, strongly, lower intermediately, and upper intermediately, less free than and freer than (respectively *WLF*, *SLF*, $T_P LF$, $T_u LF$ for less free than, and *WF*, *SF*, $T_P F$, $T_u F$ for freer than). The first two have been noted:

Definitions: equally free and no less free.

$x_I \, AF \, y_J \overset{d}{\Leftrightarrow} E_I(x_I)$ 1 $E_J(y_J) \neq \varnothing$.

$x_I \, NLF \, y_J \overset{d}{\Leftrightarrow} \exists \, X \in E_I(x_I)$ and $Y \in E_J(y_J)$: $X \supseteq Y$.

The relation *AF* is symmetrical. The following relation between these two binary relations have been noted:

Proposition 7

For x_I, $y_J \in EF$,

$x_I AF y_J \Leftrightarrow x_I NLF y_J$ and $y_J NLF x_I$.

The direct relation is obvious from the definitions, but the converse is not. The relation between these group freedom comparisons and individuals' preferences (and hence potential freedoms) will be shown to be:

Proposition 12: preference characterizations.

For $x_I, y_J \in EF$,

1) $x_I\ AF\ y_J \Leftrightarrow (i\in I, j\in J \Rightarrow x_i\ AF\ y_j)$,

2) $x_I\ NLF\ y_J \Leftrightarrow (i\in I, j\in J \Rightarrow x_i\ R_i\ y_j)$.

Proposition 12 entails proposition 7.

Definition: weakly less free and weakly freer

$x_I\ WLF\ y_J \overset{d}{\Leftrightarrow}$ no $x_I\ NLF\ y_J$,

$x_I\ WF\ y_J \overset{d}{\Leftrightarrow} x_I\ NLF\ y_J$ and no $x_I\ AF\ y_J$, or, equivalently,

$x_I\ NLF\ y_J$ and $y_J\ WLF\ x_I$.

The preference characterization of *WLF* and *WF* is a corollary of that of *NLF* in proposition 12, namely, for the former one,

$x_I\ WLF\ y_J \Leftrightarrow \exists\ i\in I$ and $j\in J$: $y_j\ P_i\ x_i$.

This leads to the introduction of three stronger concepts of less free and freer: “strongly” (*S*) and two “intermediately” (T_P and T_u where P and *u* respectively stand for lower and upper):

$x_I\ SLF\ y_J \Leftrightarrow (i\in I, j\in J \Rightarrow \mathrm{y}_j\ P_i\ x_i)$,

$x_I\ T_P LF\ y_J \Leftrightarrow [i\in I \Rightarrow (\exists j\in J$: $\mathrm{y}_j\ P_i\ x_i)]$,

$x_I\ T_u\ LF\ y_J \Leftrightarrow [j\in J \Rightarrow (\exists i\in I$: $\mathrm{y}_j\ P_i\ x_i)]$.

The relations strongly freer (*SF*) and intermediately freer are then defined as:

$x_I\ SF\ y_J \overset{d}{\Leftrightarrow} x_I\ NLF\ y_J$ and $y_J\ SLF\ x_I$,

$x_I\ T_P F\ y_J \overset{d}{\Leftrightarrow} x_I\ NLF\ y_J$ and $y_J\ T_P LF\ x_I$,

$x_I\ T_u\ F\ y_J \overset{d}{\Leftrightarrow} x_I\ NLF\ y_J$ and $y_J\ T_u\ LF\ x_I$.

The preference characterization of the four relations “freer” (*F*) result from those, just noted, of the relations *NLF* and of the four relations “less free” (*LF*).

From these definitions for both *LF* and *F* respectively, the strong relations (*S*) imply both intermediate relations (T_P and T_u) and any of these three implies the weak relations (*W*).

All the relations “freer” are antisymmetrical.

Finally, these four relations “freer” provide by negation four relations “no freer”, which are shown to be:

- for W, T_P and T_u: $x_I\ AF\ y_J$ or $x_I\ WLF\ y_J$, or, equivalently, $y_J\ NLF\ x_I$ or $x_I\ WLF\ y_J$,
- for S: $x_I\ WLF\ y_J$ or ($\exists\ i\in I$ and $j\in J$: $y_j\ R_j\ x_i$).

But other concepts of "no freer" are relevant, such as, in particular,

- $i\in I$ and $j\in J \Rightarrow x_i\ NF\ y_j$ (that is, $y_j\ P_i\ x_i$ or $y_j\ R_j\ x_i$),

or the "uniformly no freer" relation :

- $x_I\ UNF\ y_J \overset{d}{\Leftrightarrow}$ either $y_J\ NLF\ x_I$ or $y_j\ P_i\ x_i$ for all $i\in I$ and $j\in J$.

All implications between the various concepts of "no freer" are straightforward.

These concepts provide the "minimal comparisons" of freedom-ordered allocations.

5.4 Minimal comparisons with several least free individuals.

Definition: minimal comparisons.

Consider now allocations $x=\{x_i\}$ for a given population, and two freedom-ordered allocations $x, x'\in FO$. Denote as M and M' the sets of least free individuals for the allocations x and x', and as x_M and $x'_{M'}$ the corresponding suballocations of x and x', respectively. Then, the binary relations "minimally no less free, and weakly, strongly, intermediately T_P or T_u less free, freer, no freer" are defined as

$$x\ M\Phi\ x' \overset{d}{\Leftrightarrow} x_M\ \Phi\ x'_{M'},$$

where Φ can stand for NLF, WLF, SLF, $T_P\ LF$, $T_u\ LF$, WF, SF, $T_P\ F$, $T_u\ F$, and the various "no freer" concepts. The relations between these binary minimal relations directly result from the relations between the binary relations represented by these various values of Φ. The relations with "freer" are antisymmetrical. Moreover, the following properties will be shown (among others of the same type):

Proposition 13

1) If Σ stands for S or for T_P,

$x\ M\Sigma F\ x'$ *and* $i\in M' \Rightarrow x_i P_i x'_i$,

that is, the least free individuals with x' are freer with and prefer x.

2) If Σ stands for W or for T_u,

$x\ M\Sigma F\ x' \Rightarrow \exists\ i\in M' : x_i P_i x'_i$, *that is, at least one least free individual with* x' *is freer with and prefers x.*

Uniform comparisons of equal-freedom groups will denote the comparisons where all the individuals of each group are treated alike, that is, *AF*, *NLF*, *SLF*, *SF*, and *UNF*. *Uniform minimal comparisons* of freedom-ordered allocations will denote the minimal comparisons with uniform comparisons of the groups of least free individuals in each allocation, that is, *MNLF*, *MSLF*, *MSF*, and *MUNF*.

5.5 Uniform efficient freedom maximins

The various minimal comparisons give rise to general concepts of efficient freedom maximins. For a given population, we consider efficient allocations in the case where they are freedom-ordered, $x=\{x_i\}\in E$. A given relation Φ can provide two concepts which are $x\in E$ and, for each $x'\in E\setminus\{x\}$, either $x\Phi x'$ or no $x'\Phi x$. For brevity, we will solely consider uniform comparisons here. The concepts will then be analogous to those with single least free individuals. This leads to four concepts derived from the relations *NLF*, *SLF*, *SF*, and *UNF* between efficient allocations, applied to the least free groups.

These efficient freedom maximin are $x\in E$ such that for all $x'\in E\setminus\{x\}$,

1) For a *No less free efficient freedom maximin* (*NLFEFM*):

$x\ MNLF\ x' \Leftrightarrow x_M\ NLF\ x'_{M'} \Leftrightarrow (i\in M, j\in M' \Rightarrow x_i\ NLF\ x'_j \Leftrightarrow x_i R_i x'_j)$,

2) For a *Less free efficient freedom maximin* (*LFEFM*):

$x'\ MSLF\ x \Leftrightarrow x'_{M'}\ SLF\ x_M \Leftrightarrow (i\in M, j\in M' \Rightarrow x'_j\ LF\ x_i \Leftrightarrow x_i P_j x'_j)$,

3) For a *Freer efficient freedom maximin* (*FEFM*):

both $x\ MNLF\ x'$ and $x'\ MSLF\ x$,

4) For a *No freer efficient freedom maximin* (*NFEFM*):

either $x\ MNLF\ x'$ or $x'\ MSLF\ x$.

That is, no least free individual in a *NLFEFM* prefers the individual allocation of a least free individual in any efficient allocation. The individual allocation of any least free individual of a *LFEFM* is preferred to their individual allocations by all least free individuals in other efficient allocations. Both relations hold for a *FEFM*. And either relation holds for each other allocation x' for a *NFEFM*.

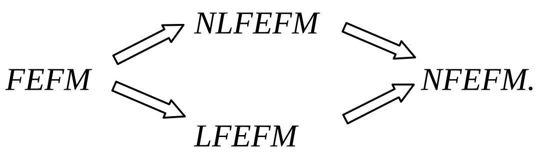
The following properties result from the definition or will be shown:

Proposition 14

1) A FEFM is unique.

2) There cannot be both a NLFEFM and a LFEFM distinct.

3) The sets of FEFM and of NFEFM are respectively the intersectioin and the union of the sets of the NLFEFM and of the LFEFM. Hence, the implications of these properties are



4) All least free individuals of all NLFEFM are equally free. All NLFEFM are minimally as free as one another.

5) Given a LFEFM, all least free individuals in other efficient allocations are freer with and prefer this LFEFM.

6) In particular, all least free individuals in efficient allocations which are not a FEFM are freer with and prefer the FEFM if there exists one.

7) In any efficient allocation which is not a NLFEFM, at least one least free individual is freer with and prefers another efficient allocation.

For the reasons stated in the single-least-free case, the maximin choice should be the *FEFM* if there is one. Otherwise, it should be another of the considered maximins if it exists, with the *NLFEFM* and the *LFEFM* having a property of uniformity in the defining relations. Moreover, in each class of maximins where there are several allocations, the second step of a concept of leximin consists of favoring the allocation(s) with the *lowest number* of least free individuals.

5.6 Freedom leximins.

If, for freedom-ordered efficient allocations, the comparison of least free individuals does not suffice to designate a maximin solution, one can then use the layer structure of freedom-ordered allocations described in section 3.3, and compare the second least free individuals, possibly among a set of already selected allocations, and so on. One can also begin with minimally least free, then extend consideration to least free with a smaller number of individuals who are freer than them, and so on.

6. DIFFERENT INDIVIDUAL POSSIBILITIES

6.1 Concepts and definitions

A given individual may not be able to have a number of individual allocations. This depends on the nature of the considered allocations. One of the most important examples occurs when an allocation includes a job or an occupation, possibly along with a wage for it, which requires particular capacities that not all the considered individuals have. Relatedly, different individuals may be able to obtain incomes with the same labor in duration, intensity, formation work, etc., which differ because they have different given capacities, and therefore their corresponding domains of possible consumption goods and labor or leisure are different. Moreover, individuals have more or less different needs for consumption which permit their survival. Individuals may also not be able to consume certain consumption goods (such as dresses of inappropriate size, items one cannot use, etc.), though in this case the inadequacy can be expresses as a low ranking in the individual's preferences. In addition, very stringent social or psychological reasons which require certain consumption or forbid others can also be treated in this manner (law, norms imposed by social pressure or by the individual's own decision, etc.). Let us also note that leisure, for instance measured in duration, may have to be considered as individual-specific, that is, each individual can benefit solely from his own and not from other's [26]. Hence, generally, for each individual i there is a domain $D_i \subseteq A$ of possible allocations for her. If individual i chooses her allocation x_i, this can solely be a $x_i \in D_i$. Hence, for describing individual i's choices, her preferences preordering R_i need be defined on the domain D_i only (preferences about allocations that one cannot have are not always well defined).

[26] See Kolm 1996a, 1996b.

From the point of view of distributive justice, this situation can give rise to two alternative ethical treatments, according as this limitation of the possible x_i to D_i is assigned to the individuals' accountability or is not, that is, as whether each individual is a priori entitled to the possibilities permitted by her D_i or is not. In the former case, the D_i are treated as individuals' preferences R_i have been. They have no direct relevance for the considered conception of public justice. They solely intervene when the criterion of equal notional freedom is expressed in the field of allocations because they determine the chosen x_i, as the R_i do. And, indeed, the domains D_i can be expressed as and reduced to a structure of the preferences R_i, in considering that individual i always prefers an alternative in D_i to an alternative not in D_i (the "sour grapes" device). In the other case, the direct evaluation of justice seeks equality in abiding by the constraints, and hence aims at correcting their inequality. Then, the D_i are constraints on the notional freedoms that are used to define equally free, no less free, and freer. These positions lead to two new definitions of the sets $F_i(x_i)$. Given these definitions, in each case the derived definitions and a number of properties are exactly as those presented above. However, a crucial difference will be met with the basic theorem deriving the freedom-order property from the absence of improving permutations, and hence its existence in efficient and core allocations: the properties hold in the case of individual accountability/entitlement for the D_i, as shown in section 3.5, but not in the other case.

If the considered conception of justice sees the fact that the possible domains D_i differ as relevant for direct justice, and hence as something which is unjust and should be corrected, F_i is to be replaced by F_i^- defined as:

$X \in F_i^-(x_i) \overset{d}{\Leftrightarrow} X \subseteq D_i$ and $x_i \in c_i(X)$,

or $F_i^-(x_i) = F_i(x_i)\ 1\ \ (D_i)$ where $\ (D_i)$ denotes the sets of parts of D_i.

If, on the contrary, the individuals are accountable for the limitations D_i of their possibilities, F_i is to be replaced by F_i^+ defined as:

$X \in F_i^+(x_i) \overset{d}{\Leftrightarrow} X 1 D_i \in F_i(x_i) \Leftrightarrow x_i \in c_i(X 1 D_i)$.

The concepts, relations and sets, *EF*, *AF*, *NLF*, and, from them, *LF*, *NAF*, *F*, *NF*, *FO*, least free, and freest, are defined from F_i^- and F_i^+ as they have been from F_i (expressions (1) to (8)). They will be distinguished with the superscripts - and + , respectively. The following relations hold:

$EF^- \subseteq EF^+$,

$AF^- \Rightarrow AF^+$,

$NLF^- \Rightarrow NLF^+$,

$LF^+ \Rightarrow LF^-$.

It will also be shown that:

Proposition 15

Proposition 1 holds for each of the two new definitions of equal freedom EF^+ *and* EF^-.

The same holds for proposition 2 that is,

Proposition 16

$$x_i\ AF^- x_j \Leftrightarrow x_i\ NLF^- x_j \text{ and } x_j\ NLF^- x_i,$$
$$x_i\ AF^+ x_j \Leftrightarrow x_i\ NLF^+ x_j \text{ and } x_j\ NLF^+ x_i\,.$$

6.2 Characterizations from preferences and possibilities

All these relations can be characterized from individuals' preferences and possibilities. A relation $x_j \in D_i$ means that individual *i* can have individual *j*'s allocation or can take individual *j*'s place. The following properties will be shown:

Proposition 17

17-1: $x_I \in EF^- \Leftrightarrow (i \in I \Rightarrow x_i \in 1_{j\in I} D_j)$ *and* $(i,j \in I \Rightarrow x_i R_i x_j)$,

each individual can have each other's allocation but does not prefer it to her own.

17-2: $x_I \in EF^+ \Leftrightarrow (i,j \in I \Rightarrow x_i R_i x_j$ *or* $x_j \neq D_i)$,

each individual does not prefer each other's allocation to her own or cannot have it, that is, each individual does not prefer each other's allocation that she can have, or cannot have each other's allocation that she prefers.

17-3: $x_i \ NLF^- x_j \Leftrightarrow x_i R_i x_j$ *and* $x_j \in D_i$,

individual i *can have but does not prefer individual* j*'s allocation.*

17-4: $x_i \ NLF^+ x_j \Leftrightarrow x_i R_i x_j$ *or* $x_j \notin D_i$,

individual i *does not prefer or cannot have individual* j*'s allocation, or she does not prefer it if she can have it, or she cannot have it if she prefers it.*

Corollaries

$x_i \ NLF^+ x_j \Leftrightarrow x_i \ NLF^- x_j$ or $x_j \notin D_i$.

This confirms $NLF^- \Rightarrow NLF^+$.

$x_i \ LF^- x_j \Leftrightarrow x_j P_i x_i$ or $x_j \notin D_i$,

individual *i* prefers or cannot have individual *j*'s allocation, or she prefers it if she can have it, or she cannot have it if she does not prefer it.

$x_i \ LF^+ x_j \Leftrightarrow x_j P_i x_i$ and $x_j \in D_i$,

individual *i* prefers and can have individual *j*'s allocation.

$x_i \ LF^- x_j \Leftrightarrow x_i \ LF^+ x_j$ or $x_j \notin D_i$.

This confirms $LF^+ \Rightarrow LF^-$.

$x_i \ F^- x_j \Leftrightarrow x_i R_i x_j$, $x_j \in D_i$, and ($x_i P_j x_j$ or $x_i \notin D_j$) .

$\mathrm{x_i} \ F^+ \mathrm{x_j} \Leftrightarrow \mathrm{x_i\ P_j\ x_j}$, $\mathrm{x_i \in D_j}$, and ($\mathrm{x_i\ R_i\ x_j}$ or $\mathrm{x_j \notin D_i}$) .

6.3 Possibilities with various accountabilities

More generally, the various possibilities and impossibilities or constraints can have different statuses of accountability. Then, the allocation of an individual *i* is restricted by two domains of possibilities, D_i^+ for which she is accountable, and D_i^- for which public justice is accountable, with $D_i^+, D_i^- \subseteq A$, and the overall constraint $x_i \in D_i^+ 1 D_i^-$. Then, the relevant set of possible potential freedoms becomes F_i^m, which replaces F_i (the superscript *m* stands for "mixed") defined by

$$X \in F_i^m(x_i) \overset{d}{\Leftrightarrow} X \subseteq D_i^- \text{ and } x_i \in c_i(X 1 D_i^+) .$$

All the concepts of the theory can then be derived from this expression, with their relations and their characterizations in terms of preferences and domains of possibilities.

6.4 Comparison of an individual's freedoms, efficiency, and its realistic structure.

Many properties presented without specifications of the D_i extend rather straightforwardly to the cases where such constraints are present and differ across individuals, with the various moral treatments of these constraints and possibilities. In addition to equal freedom just noted, this is the case for freedom-ordered allocations and all their structural properties.

Moreover, the comparisons of the (potential) freedoms offered to one individual by two allocations, say x_i and $x_i^{'}$ for individual i, are derived from the general freedom comparisons of x_i and x_j in considering that individual j is individual i and in writing $x_j = x_i^{'}$. We then have $R_j = R_i$ and $D_j = D_i$. Proceeding to these substitutions in proposition 17 and its corollaries provides the expressions of the freedom comparisons for the same individual in terms of her preferences and individual possibilities. If we restrict consideration to possible allocations in assuming a priori x_i and $x_i^{'} \in D_i$, the result turns out to be that the freedom comparisons for one individual coincide with her preference comparisons: AF is I_i, NLF is R_i, F is P_i, and LF is P_i in reverse. This can be summarized as:

Proposition 18
For an individual and her allocations permitted by her own personal possibilities, whatever the entitlement/accountability status of these possibilities, the (potential) freedom and the preference comparisons coincide.

One consequence is that (Pareto-)efficiency can be expressed with these freedom comparisons as with preferences as it classically is.

However, there is a crucial difference between the various ethical treatments of the D_i for one crucial property, the possibility of actual improving permutations and hence the freedom-order structure of efficient and core allocations.

Indeed, the individual possibilities D_i are among the constraints which define efficiency and limit or permit individual or collective actions that define the core. Hence, the relevant improving permutations are such that involved individuals receive individual allocations that they both prefer and can have: individual i receives x_j such that $x_j P_i x_i$ and $x_j \in D_i$. The corollaries of proposition 17 show that this is $x_i\, LF^+ x_j$. Hence, the relevant cycles of relations "least free" are with LF^+, and therefore the ethical treatment of the D_i which leads to the freedom-order property of efficient (and core) allocations is individual self accountability and entitlement (see section 3.5).

In the other case, by contrast, least free is LF^-, which is $x_i\, LF^- x_j \Leftrightarrow x_j P_i x_i$ or $x_j \notin D_i$ (corollary of proposition 17). Hence this relation can be satisfied by $x_j \notin D_i$, that is, individual i cannot have allocation x_j, and in this case the corresponding permutation is not actually possible.

Hence, the freedom-order property of efficient (and core) allocations when individual possibilities differ requires the former case, that of self entitlement/accountability of individual possibilities and limits (in addition to the other assumptions – see proposition 6). The corresponding equal-freedom is the classical "realistic equity" (no one prefers any other's allocation that she can have)[27]. "Realistic" refers to the taking of individual's possibilities and limits into account, and this qualificative will be kept for this case. From proposition 6 and the properties of the freedom-order structure, which are valid for this case, all the analysis of the maximin concepts, with their definitions, relations, and properties, remain valid for this "realistic" case. Most definitions and properties in fact solely use the freedom comparisons. When preferences are written, then $x_i R_i x_j$ is to be replaced by $x_i\, NLF^+ x_j \Leftrightarrow x_i R_i x_j$ or $x_j \notin D_i$, and $x_j P_i x_i$ is to be replaced by $x_i\, LF^+ x_j \Leftrightarrow x_j P_i x_i$ and $x_j \in D_i$.

For all these properties, in fact, the realistic case is the general case since the other corresponds to the particular case where domains D_i are identical or absent. Conversely, though, the realistic case can formally be reduced to the case with no D_i in using the "sour grapes" preferences $R_i^{'}$ derived from R_i in assuming that the individual never prefers an allocation she cannot have (see section 1.3).

6.5 Realizations and the structure of possibilities

[27] See Kolm 1971.

Assume the individual possibilities D_i are the only constraints on the allocation. If the D_i are identical, efficient equal-freedom allocations can be achieved by simply letting the individuals freely choose their own allocation. The entitlement/accountability status of the D_i makes no difference. In all cases, a domain identical to these D_i can be the equal-freedom domain X.

If the D_i differ across individuals, letting the individuals freely choose their allocation in their own D_i provides an allocation which is efficient and realistically equal-freedom. In this choice, indeed, each individual either cannot have or does not prefer any alternative other than her choice, and this can apply to any other individual's allocation. For instance, process-liberalism is often described as free action or free exchange but, as regards distribution, it means entitlement to the full outcome of one's such action, and, in fact, to the domain of choice defined by one's own capacities and initial endowments. Then, in the conditions in which process-freedom is efficient, such as perfect competition in which individuals' domains of choice are de facto independent (parametric prices), it is efficient and realistically equal-freedom. The same holds for equal labor income equalization if the entitlements to rents in others' capacities during the equalization labor are considered legitimate[28].

With this self entitlement/accountability for the individual domains of possibilities D_i, a notional domain of choice can be any X including all D_i , $X \supseteq \cup D_i$. It always exists. It can for instance be the union of these domains, $X = \cup D_i$. By contrast, with the opposite moral assumption, the common notional domain of choice should satisfy $X \subseteq \cap D_i$; it can for instance be the intersection of these domains $X = \cap D_i$. It does not always exist. And when it exists, individuals' free choices in X do not generally provide an efficient allocation (they can provide an efficient allocation if and only if $c_i(D_i) \cap (\cap D_i) \neq \emptyset$ for all i).

However, in a more general situation, what an individual can have depends on what the others' have. The constraints are not separable, and individuals' freedoms are not independent, as they are with the considered D_i . Or, more generally, such constraints exist in addition to individualizable ones of the type of the D_i , as considered in section 3.5. Then,

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[28] See Kolm 1996b, 1998.

with individuals' entitlement for the D_i (the "realistic" case), solely these extra constraints need be considered for the normative consequences of the possibilities.

Proposition 6, and hence the freedom-ordered structure of efficient and core allocations and all the theory of the corresponding maximins, hold when these extra constraints and domain are symmetrical in the individual allocations, that is, when they allow all permutations of the individual allocations. This case has a fairly large domain of validity. It means that the considered agents have in some sense the same characteristics for "occupying the world", including for the various interrelations between them. This is in particular the general case for the allocation of resources of various types. Moreover, these extra constraints can often be more or less controlled by policy rules, and this symmetry-permutability also has an ethical dimension which may require its establishment. Indeed, it constitutes the property of equal (identical) freedom in the case of interferring liberties. Interferring liberties means that an agent's freedom of action or choice depends on others' action of choice (the cases where this does not hold are those of independent freedoms, as with the D_i considered above). Then, the symmetry of the possibility set has been shown to be identical to the principle of *equal liberty potential*, defined as the condition: "if you did what I do, I could do what you can do"[29]. If this describes actual actions which are sequential in time, this principle is equality with an entitlement/accountability for the effects on oneself of the dates or order of actions – a kind of "right of first occupancy". Note that symmetry-permutability for independent freedoms is the identity of these domains of choice.

7. PROOFS

Proof of proposition 8

Let $\{x_I\}$ denote the set of the $|I|$ nonordered and unassigned individual allocations x_i for $i \in I$.

Proof of 8-5: $x_I \in EF \Leftrightarrow (i,j \in I \Rightarrow x_i R_i x_j)$.

[29] Kolm, 1993.

$x_I \in EF \overset{d}{\Rightarrow} \exists\ X$: $X \in F_i(x_i)$ for all $i \in I$, and hence $x_i \in c_i(X)$ for all $i \in I$, and therefore $x_i R_i x_j$ for all $i,j \in I$ since $x_j \in X$.

Conversely, if $x_i R_i x_j$ for all $i,j \in I$, then $\{x_I\} \in F_i(x_i)$ for all $i \in I$, and hence $E_I(x_J) \neq \emptyset$.

Proof of 8-2: $x_i\ NLF\ x_j \Leftrightarrow x_i R_i x_j$.

$x_i\ NLF x_j \overset{d}{\Rightarrow} \exists\ X \in F_i(x_i)$, $Y \in F_j(x_j)$: $X \supseteq Y$. But $x_i \in c_i(X)$ and $x_j \in Y$ and hence $x_j \in X$. Thus $x_i R_i x_j$.

Conversely, if $x_i R_i x_j$, $X=\{x_i,x_j\}$ and $Y=\{x_j\}$ are such that $x_i \in c_i(X)$ and hence $X \in F_i(x_i)$, $Y \in F_j(x_j)$, and $X \supseteq Y$, and hence $x_i\ NLF\ x_j$.

These two results entail the other parts of proposition 8, and propositions 1 and 2.

Proof of proposition 17

Proof of 17-2: $x_I \in EF^+ \Leftrightarrow (i,j \in I \Rightarrow x_i R_i x_j$ or $x_j \notin D_i)$.

$x_I \in EF^+ \Rightarrow \exists\ X \in F_i^+(x_i)$ for all $i \in I$, and hence $x_i \in c_i(X \cap D_i)$ for all $i \in I$. Thus, $x_j \in X \cap D_j$ and $x_j \in X$ for all $j \in I$. Therefore, if $x_j \in D_i$, then $x_j \in X \cap D_i$, and $x_i R_i x_j$.

Conversely, $x_i R_i x_j$ or $x_j \notin D_i$ for all $i,j \in D_i$ implies $x_i R_i x_j$ if $x_j \in D_i$ for all $j \in I$, and hence $x_i = c_i(\{x_I\} \cap D_i)$, that is $\{x_I\} \in F_i^+(x_i)$, for each $i \in I$.

Proof of 17-4: $x_i NLF^+ x_j \Leftrightarrow x_i R_i x_j$ or $x_j \notin D_i$.

$x_i NLF^+ x_j \overset{d}{\Rightarrow} \exists\ X \in F_i^+(x_i)$, $Y \in F_j^+(x_j)$: $X \supseteq Y$. This implies $x_i = c_i(X \cap D_i)$, and $x_j \in Y \cap D_j$ and hence $x_j \in Y$ and $x_j \in X$. Thus, if $x_j \in D_i$, then $x_j \in X \cap D_i$, and hence $x_i R_i x_j$.

Conversely, $x_j \in D_i$ and $x_i R_i x_j \Rightarrow x_i \in c_i(\{x_i,x_j\} \cap D_i)$. And $x_j \in c_j(\{x_j\} \cap D_j)$. Thus the sets $X=\{x_i,x_j\}$ and $Y=\{x_j\}$ satisfy the conditions $X \in F_i^+(x_i)$, $Y \in F_j^+(x_j)$, and $X \supseteq Y$, and therefore $x_i\ NLF^+ x_j$.

The proofs of propositions 18-1 and 18-3 are analogous and left for the reader.

These results entail the corollaries of proposition 18 and propositions 15 and 16.

Proof of proposition 12

If $x_I AF y_J$, there is a nonempty domain $X \in E_I(x_I) \cap E_J(y_J)$. Hence for all $i \in I$, $j \in J$, $X \in F_i(x_i) \cap F_j(y_j)$ and $x_i AF y_j$. Conversely, if $x_i AF y_j$ for all $i \in I$, $j \in J$, and since x_I, $y_J \in EF$, $X = \{x_I\} \chi \{y_J\} \in EF$, and $X \in E_I(x_I) \cap E_J(y_J)$.

If $x_I NLF y_J$, $\exists X \in E_I(x_I)$ and $Y \in E_J(y_J)$: $X \supseteq Y$. Hence, $y_j \in Y$ and $y_j \in X$ for all $j \in J$, and $x_i \in c_i(X)$ for all $i \in I$, and hence $x_i R_i x_j$ for all $i \in I$ and $j \in J$. Conversely, $X = \{x_I\} \chi \{y_J\}$ and $Y = \{y_J\}$ are such that $X \supseteq Y$, $Y \in E_J(y_J)$ since $y_J \in EF$, and $X \in E_I(x_I)$ since $x_I \in EF$ and if $x_i R_i y_j$ for all $i \in I$ and $j \in J$.

Proof of proposition 3

If $i \ \ j \Rightarrow (\exists\, X_i \in F_i(x_i), X_j \in F_j(x_j): X_i \supseteq X_j)$, then $x_j \in X_j$, $x_j \in X_i$, and hence $x_i R_i x_j$.

If $i \ \ j \Rightarrow x_i R_i x_j$, then $X_i \overset{d}{=} \{x_j\}_{j \ \ i} \in F_i(x_i)$, with $j \ \ i \Rightarrow x_j \in X_i$ and $j \ \ i \Rightarrow X_j \subseteq X_i$, where $j \ \ i$ means $j \ \ i$ or $j = i$.

Proof of proposition 4, and section 6.3

The proof of proposition 4 is practically implied by its enunciation. The freedom ranking of an allocation implies the existence of least free and of freest individuals. Then, remove all individuals who are not both least free and freest, along with their individual allocations. Nothing remains if the initial allocation is equal-freedom, but there remains a nonempty suballocation if it is not equal-freedom. This suballocation also is freedom-ordered. Its freest individuals are no less free than all others in the suballocation, and also in the initial allocation since the removed individuals are least free in this initial allocation. They thus also are freest in the initial allocation. And they are not also least free in this initial allocation since, otherwise, they would have been removed. An analogous reasoning applies for the least free individuals and proves the theorem.

The layer structure of freedom-ordered allocations, shown in section 6.3, also implies proposition 4 and provides further properties of these allocations.

Proof of proposition 5.

Proposition 5-1

Consider a freedom-ordered allocation with “ ” denoting the corresponding strict ordering of indices $i,j,k,\ldots$ That is, i $j \Rightarrow x_iNLx_j$. Hence, $x_iLFx_j \Rightarrow i$ j. Thus, if there were a cycle $i,j,k,\ldots i$ such that $x_iLFx_jLFx_k\ldots LFx_i$, one would have i j k i, and hence i i since denotes a strict ordering, which is impossible for the same reason. Therefore, the freedom ranking structure bans less free cycles.

Proposition 5-2

Assume there is no less free cycle for the allocation x_I. Denote as $\hat{F}(x_I)$ and $\widehat{LF}(x_I)$ the sets of freest and least free individuals of population I with allocation x_I . A priori, they may be empty). Consider any $i\in I$.

If $i\in \hat{F}(x_I)$, then $\hat{F}(x_I) \neq \emptyset$. If $i\notin \hat{F}(x_I)$ there exists $j\in I$ such that x_iLFx_j. If $x_j\in \hat{F}(x_I)$, then $\hat{F}(x_I) \neq \emptyset$. If $x_j\notin \hat{F}(x_I)$, there exists $k\in I$ such that x_jLFx_k. And so on. Either there is an end with a $x_P\in \hat{F}(x_I)$, or this is not the case. If the individuals are in finite number, in the latter case at one point an individual who has already been considered has to be met. Then, a closed loop of individuals including this individual has been followed, as part (or all) of the whole path. Along this loop, each individual is less free than the next: they constitute a less free cycle. Thus, with a finite number of individuals ($|I|<\infty$), the absence of less free cycles implies $\hat{F}(x_I) \neq \emptyset$. Delete now the freest individuals in considering the subset of individuals $I' = I \setminus \hat{F}(x_I)$ and the corresponding suballocation $x_{I'}$. If $|I|<\infty$, then $|I'|<\infty$. A least free cycle with $x_{I'}$ also is one with x_I, and hence the absence of less free cycle with x_I implies that of less free cycles with $x_{I'}$. Repeat, then, with $x_{I'}$, the same reasoning as the foregoing one with x_I . Then, there is a nonempty set of freest individuals in $x_{I'}$, $\hat{F}(x_{I'}) \neq \emptyset$. Delete these from I', and continue similarly. Since $|I|<\infty$, this process has an end. We thus have constructed a hierarchy of layers $\hat{F}(x_I)$, $\hat{F}(x_{I'})$, $\hat{F}(x_{I''})$,… such that each individual of a layer is no less free than all the individuals of further layers and as free as those of her layer. This shows that x_I is a freedom-ordered allocation.

An analogous reasoning, but considering the least free individuals rather than the freest, ends up to the same conclusion. Then, if, for any $i \in I$, $x_i \notin \hat{LF}(x_I)$, there exists a $j \in I$ such that $x_j LF x_i$. If $x_j \notin \hat{LF}(x_I)$, one continues. If this could go on indefinitely, $|I|<\infty$ would imply the existence of a less free cycle. Hence $\hat{LF}(x_I) \neq \emptyset$. Reproduce the reasoning with the subset of individuals $I' = I \setminus \hat{LF}(x_I)$ and so on. A freedom-ordered structure for x_I thus is obtained as $\ldots, \hat{LF}(x_{I'}), \hat{LF}(x_I)$.

QED.

Varian's (1976) remark that in a "fair" (equitable and efficient) allocation of bundles of commodities one individual "envies" no other and one individual is "envied" by no other is a consequence of proposition 6 since it is a consequence of the freedom order of the allocation, under the conditions (which were not explicit) of proposition 6: a finite number of individuals and unlimited permutability – or, alternatively, permutations solely limited by individuals' limitations for which they are held accountable and replacement of "envies" by "either 'envies' the other's allocation or cannot have it". Varian also suggests that this "unenvied" individual could be used in a maximin.

Proof of minimal comparisons.

Propositions 10,11,13, and 14 include properties stating that if $x' MLF x$, and $i \in M'$, then $x_i' LF x_i$ (and $x_i P_i x_i'$). Indeed, if $j \in M$, $x_i R_i x_j$ from the definition of M, and $x_j P_i x_i'$ from the definition of MLF. Hence $x_i P_i x_i'$.

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