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ETHICAL ECONOMIC INEQUALITIES

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1. Introduction

People die in revolutions fighting inequality. Inequalities arouse emotions of moral indignation and judgments of moral indictment or reproof. They are then equated with injustice. Aristotle already noticed this: “Justice is equality, as everybody believes it to be, quite apart from any other consideration” (*Nicomachean Ethics* and *Eudemian Ethics*). Since economics is the study of the allocation of goods to people, normative economics and economic ethics are largely the determination of the relevant equalities and therefore, when these equalities are not achieved, the comparison of the corresponding inequalities. This analysis is also a main chapter of mathematical social ethics.

Even when inequality is a purely descriptive concept meaning dispersion, it can have moral implications. Poverty of all degrees results from inequality. It may arouse sentiments of compassion and pity, and elicit acts of charity and solidarity, which are morally praised. However, inequality also induces morally condemned social sentiments such as envy (“the most odious of social sentiments” in the words of J.S. Mill), jealousy, or sentiments of superiority. Moreover, particular structures of inequality imply or are related to various social structures which are important in moral evaluations of society, such as hierarchical structures, polarization into possibly hostile groups, class or cast structures, isolation of groups of poor or rich, etc. However, we will essentially focus here simply on inequality as injustice.

Strangely for a feeling which is moral and can be intense, the sentiment of the injustice of inequality has a purely logical and rational basis. The problem is the lack of a valid reason or justification for the differences in evaluated situations. Justice should be justified, rational in the most common and basic sense of the term of providing a reason or intending to (since no reason is ever complete: one can always ask a further question “why?”). Hence, if a given allocation of something to someone is justified in using a set of characteristics of this person, then, *prima facie,* some other person who has the same set of these relevant characteristics should have the same relevant allocation. When this equality is not achieved and one person receives a better treatment than the other in some relevant sense, there is a corresponding situation and sentiment of injustice, which can be smaller or larger. Of course, the relevant equality is sometimes impossible for various possible reasons (indivisibility, non-transferable characteristics, various social and political facts), or it can oppose some other value jointly retained (possibly the ideal equality of some other items, or some unequal state provides more to everyone). This question, "why equality," has logical precedence over the question "equality of what." It has further basic developments which cannot be presented here.[1] However, the moral value or indignation derives not only from the rational *prima facie* necessity of equality, but jointly from the tangible characteristics of the ideal equality: of what, between whom, where and when?

At any rate, inequalities deemed relevant constitute a major issue for judging societies and policies. They are often compared, for instance across countries or over time. This is in particular commonly done by the media and politicians. However, this presentation is almost always highly puzzling and problematic. Indeed, given any two unequal distributions of some item, one can most of the time show that anyone is more unequal than the other and the converse, with arguments, reasons, comparisons and measures which, a priori, seem all convincing.

Examples can show how this is easily done. The effect of growth on inequality is one of the most common topics in economics. Balanced growth in which incomes grow more or less at the same rate tends to leave income inequalities based on ratios of incomes unchanged, while it tends to augment discrepancies between incomes and hence inequalities based on differences.[2] Yet, in democratic countries, policy tends to redistribute part of the fruits of growth, but not to the point of equally sharing them (which may undercut incentives). In the end, inequalities based on ratios tend to decrease, whereas inequalities based on differences still tend to increase. Whatever the facts, the question of the effects of growth on inequality is meaningless as long as you do not specify what you mean by inequality. You may "believe" in inequalities based on ratios. This implies in particular that, for two persons, inequality is not changed by passing from 0.01 and 1 to 0.1 and 10 (for instance from Burkina-Faso to New

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[1] See Kolm 1996 (Chapter 2), 1998 (New Foreword, Section 5), and 2004 (Chapter 23).
[2] We will see that the terms "absolute" and "relative" inequality can lead to confusion.

York city). This is by no means obvious. Or, perhaps, you “believe” in inequalities based on differences. This would imply in particular that inequality is the same with 1 and 2 and with 11 and 12 (or 1001 and 1002).

In another example, Australia seems to be a much more egalitarian country than France which has a much larger number of different incomes (even adjusting for population size). Australia has a large relatively egalitarian middle class which leads to this impression and result. But it also has a class of very rich and other people quite poor. In spite of this impression and result, one can pass from the Australian distribution to the French one (adjusted for population) by a series of income transfers each from a richer to a poorer. Could one country be less unequal and the other more just?

In fact, does such a transfer from richer to poorer diminish inequality? If a transfer of 1 from someone who has 4 to someone who has 1 certainly diminishes the pairwise inequality between these two persons, when applied in a society of four persons in which two have 1 and two have 4, thus transforming the distribution from (1,1,4,4) to (1,2,3,4), it also breaks down two strict equalities. More generally, a transfer from a richer person to a poorer one certainly increases the pairwise inequalities between the poorer and the still poorer and equally poor, and between the richer and the still richer and equally rich. Does the goodness of a change of a distribution imply an increase in the goodness of the distribution? A similar question can be asked for the increase in a single high income, which augments the inequality. More specifically, these are instances of the general question: are the unchanging incomes relevant when evaluating the goodness of the change or the change in the goodness?

In conclusion, the full and useful analysis of inequality has to be more refined and developed. The noted properties belong to the essential building blocks of the discussion but are only a beginning. The proper comparisons or measures of inequality depend on what one wants to do with them. To find the relevant ethical views, one may have to investigate them in a population (this may also be required by democracy). The results are commonly surprising on two grounds. First, sometimes some people approve a property while seeing no reason for some others, other people approve some of the latter properties while seeing no reason for the former one, while all these properties turn out to be mathematically equivalent. Second, people often have a number of common opinions which are rather subtle, as they discriminate according to the various aspects of the situation. In the end, there will have to be intermediate properties and specific domains of relevance of properties. The resulting criteria or measures will a priori depend on properties of the actual problem, such as the nature of the items, the population, and the magnitude and causes of the transformation.

Since distribution, notably of incomes, is one of the main topics of economics, economists have very early been concerned with inequalities. They developed measures of inequality (e.g. Gini 1955, Theil 1967), comparisons of distributions (e.g. Lorenz 1905), consideration of the effects of transfers from richer to poorer (e.g. Pigou 1912, Dalton 1920), or of changes of all incomes in the same proportion (Dalton 1920, Taussig 1939) or by the same amount (Dalton 1920, Cannan 1930, Loria 1934). Yet, these studies were handicapped by their lack of distinction between the two meanings of inequality – dispersion and injustice.

After such scattered occasional and casual remarks about dispersion with a vague feeling of injustice, the rational ethical analysis of unjust inequality was opened by two remarks. First, *the effects of inequality and notably injustice are among the reasons for the overall ethical evaluation of a distribution, and they can be measured by the cost of inequality implied by this evaluation* (Section 2). Second, *a number of important basic properties of the comparison of inequalities happen to be logically equivalent*, thus providing the basis of the modern ethico-logical analysis of economic inequalities (Section 4).

2. The measures of economic inequality derived from overall ethical evaluations[3]

2.1 Overall ethical evaluation and inequality

The cost of inequality, notably of its injustice, is implicit in any overall ethical evaluation, and therefore its measure can be derived from this evaluation. However, the converse view is also relevant. An overall ethical evaluation is the synthesis of moral judgments about the various relevant aspects and properties of the situation, and one or some of these aspects or properties can concern inequality and in particular distributive injustice. Then, direct moral judgments about inequality matter. The overall judgment aggregates the various particular ones in a way that has to respect properties of consistency. The other relevant judgments, and the moral

[3] The rest of this presentation of the basic properties of economic inequality consists of a simplified presentation of Sections 6 and 7 of the article *The Optimal Production of Social Justice* (Kolm 1966).

synthesis, depend on the case, and the main ones will be noted. A central point consists in the relations between the properties properly related to inequality and the other properties of the ethical judgment about the distribution.

Consider again the simplest and important case of the distribution of incomes – or any other desired quantity (other cases will be noted shortly). There are n individuals indexed by i=1, ...n. Denote as x_i the income of individual i. A distribution is a set of n x_i, one for each individual i. Such a distribution is *equal* when all the x_i are equal. The sum $X=\Sigma x_i$ is the *total* or *social income*. The *average income* is $\bar{x}=X/n$. For an equal distribution, $x_i=\bar{x}$ for all i.

The overall ethical evaluation need only be by judgments of better or worse. It is described by an ethical evaluation function $W(x_1,\ldots x_n)$ which takes a higher value when the distribution is considered to be better. The *nature* of this function is not further specified, and, hence, this function can be replaced by any increasing function of itself; that is, it is ordinal, and any increasing function of it is one of its specifications.

Moreover, we assume that the situation improves if one income increases while no other decreases, that is, the property of *benevolence*. This translates as function W being an increasing function of the x_i.

Finally, the present concern about the ethics of inequality leads us to assume that all judgments relevant here about how to share a given total income X can be expressed through judgments about the inequality of the distribution.

2.2 The equal equivalent income

For the overall evaluation, a distribution can be replaced by any other that gives the same level to function W. In particular, it can be replaced by one such distribution which is also equal. The individual income of this latter distribution is called the *equal equivalent income* of the initial distribution, and it is classically denoted as $\bar{\bar{x}}$. It is therefore defined by the equality

$$W(x_1,\ldots x_n)= W(\bar{\bar{x}},\ldots \bar{\bar{x}}). \qquad (1)$$

This level $\bar{\bar{x}}$ is uniquely defined because function W is increasing (benevolence). Hence, *the equal equivalent income of a distribution is the individual income of the equivalent equal* *distribution*. It is the individual income such that, if all individuals had it, the resulting equal distribution would be as good as the distribution in question.

The equal equivalent income $\bar{\bar{x}}$ is a function of the distribution ($x_1,\ldots x_n$) and of the function W (it is a "functional" of function W). The expression $W= W(\bar{\bar{x}}, \ldots \bar{\bar{x}})$ shows that it is an increasing function of the value or level W. Hence, it is a particular specification of this ordinal evaluation function. Moreover, it has the nature of an individual income.

If the initial distribution ($x_1,\ldots x_n$) is equal, $x_i= \bar{x}$ for all i, and hence, from equation (1), $\bar{\bar{x}} = \bar{x}$. If the evaluation function W has a specification of the form Σx_i, equation (1) writes $\Sigma x_i = n\bar{\bar{x}}$, and hence $\bar{\bar{x}} = \bar{x}$ again. This form of W implies that the ethical evaluation resents no injustice in any inequality resulting from the distribution of a total income $X=\Sigma x_i=n\bar{x}$.

2.3 The basic ethically derived indexes

If the inequality in the distribution ($x_1, \ldots x_n$) is morally bad, in particular unjust, this implies that the equal sharing of the total income $X=\sum x_i$, the equal distribution $(\bar{x},\ldots\bar{x})$, is better, that is

$$W(x_1,\ldots x_n) < W(\bar{x},\ldots\bar{x}). \tag{2}$$

A discrepancy between these two values of function W measures a moral cost of inequality. Note that $\bar{x}$ is the equal equivalent income of the equal distribution $(\bar{x},\ldots\bar{x})$. Inequality (2) also writes, given definition (1),

$$W(\bar{\bar{x}},\ldots\bar{\bar{x}}) < W(\bar{x},\ldots\bar{x}),$$

which implies $\bar{\bar{x}} < \bar{x}$. A cost is a difference between two values. Since function W is ordinal, a difference (or a ratio) in values of W is a priori not meaningful with respect to this property. However, the operation of difference (and ratio) is meaningful between quantities. It is, therefore, for the specification of W that is the equal equivalent $\bar{\bar{x}}$. Hence, the difference $\bar{x} - \bar{\bar{x}}$ is a cost in income term of the inequality of distribution ($x_1, \ldots x_n$). However, the cost can also be expressed in relative terms, by ratios, or for the whole population, as expressed by the following six classical meaningful indexes:

$I^a = \bar{x} - \bar{\bar{x}}$: *absolute* (per person) *inequality*.

$I^t = nI^a = X - n\bar{\bar{x}}$: *total inequality*.

$I^r = I^a / \bar{x} = I^t / X = 1 - (\bar{\bar{x}} / \bar{x})$: *income relative inequality*.

$I^e = I^a / \bar{\bar{x}} = I^t / n\bar{\bar{x}} = (\bar{x} / \bar{\bar{x}}) - 1$: *equal equivalent income relative inequality*.

$\eta = \bar{\bar{x}} / \bar{x} = 1 - I^r$: the *equal equivalent yield* of the distribution.

$\gamma = \bar{x} / \bar{\bar{x}} = 1 + I^e$: the *unit cost of the equal equivalent income*.

Each of these six indexes is meaningful and is the relevant one for specific questions met in the theoretical and applied analyses of inequality.

If the distribution is equal, or if Σx_i is a specification of the evaluation function *W*, $\bar{\bar{x}} = \bar{x}$, $I^a = I^t = I^r = I^e = 0$, and η=γ=1. With an unequal distribution and a cost of inequality, $\bar{\bar{x}} < \bar{x}$, I^a>0, I^t>0, I^r>0, I^e>0, η<1, γ>1. With an extreme inequality-aversion, the smallest of the x_i, $\min_i x_i$, is a specification of function *W*, then $\bar{\bar{x}} = \min_i x_i$, and the six indexes have the corresponding values.[4] For the general function *W*, each index is in between these two limiting values.[5,6]

3. Elementary properties

When the evaluation function *W* has a certain structure[7] – which is in particular satisfied if it has specifications of the form $\Sigma f(x_i)$ where function *f* is increasing and concave (it increases less and less when x_i increases by successive equal amounts) –, the foregoing ethical evaluation-consistent measures of inequality classify distributions according to a comparison which has a number of other remarkable properties, such as: a transfer from a richer person to a poorer one diminishes inequality, the Lorenz curve of a distribution of a given total income is above that of another, and a number of other meaningful ways to compare inequalities. Before showing these properties, let us note a few more elementary properties that will be used.

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[4] This particular *W* is no longer strictly increasing in all its arguments.
[5] Further concepts have been defined when the overall evaluation is such that, for some distributions, $\bar{\bar{x}} > \bar{x}$.
[6] In a didactic and influential article, Atkinson (1970) also considered the equal equivalent income $\bar{\bar{x}}$ (the “equally distributed equivalent income”) and the relative measure I^r.
[7] Presented in Section 3.2.

A distribution to two persons (n=2) (x'_1, x'_2) is *inclusion more equal* (more equal by inclusion) than another (x_1, x_2) if x'_1 and x'_2 are in between x_1 and x_2, with the possibility that x'_1 or x'_2 is equal to x_1 or x_2 if the other is not also equal to the other x_1 or x_2 (a strict inclusion of the segments between the two incomes).

Comparisons are *constant-sum* when they compare distributions with the same total X or average $\bar{x}$ (for a given number n).

If, when only a subset of the x_i changes, a comparison of the distributions does not depend on the levels of the other, unchanging x_i, this comparison for n>2 is said to be *independent* (or separable). Independence for the overall evaluation occurs if and only if a specification of the ordinal function W has the additive form $\Sigma f_i(x_i)$.[8]

If the incomes x_i are the only characteristics that relevantly differentiate the individuals for the problem at hand, the comparisons or measures are unchanged if the x_i are permuted (*invariance under permutations*). The corresponding functions – such as W – are *symmetrical* (by definition of the term). Note that this implies in particular that peoples' different specific tastes, needs, utilities, other possibilities, etc. are found not to be relevant. In particular, W cannot be a classical social welfare function depending on individuals' utilities since individuals' utility functions are a priori different.[9] If it means "welfare," this is welfare evaluated otherwise, by a judgment not following the individuals' evaluations of their own welfare, and the meaning of this concept has to be explained (which has not been done yet). However, we will consider a property that holds for all such judgments having some general properties. This symmetry is assumed in this simple presentation, but the cases in which it is not relevant have been studied. Symmetry plus independence of the function W hold if and only if it has a specification of the form $\Sigma f(x_i)$.

4. The core moral logic of economic inequalities

4.1 The basic ethical comparisons of economic inequalities

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[8] It suffices that the independence property holds when only a properly chosen set of subsets of the x_i changes, which can be reduced to n!1 subsets, or to all pairs of x_i, or to n!1 chosen pairs.

[9] Justifying the symmetry from such a function by a lack of information about individual utilities is possible but analytically delicate (Kolm 1999).

4.1.1 The transfer principle

A *progressive transfer* is a transfer from a higher income to a lower one of less than the difference (or not higher than half the difference). The *transfer principle* proposes that a progressive transfer diminishes inequality.

The transfer principle can be justified by the assumption that the unchanged incomes are irrelevant for the comparison and, given that it maintains the total sum constant, either the fact that it inclusion-reduces the inequality between the changing incomes, or the assumption that the increase in the poorer person's "welfare" overcompensates the decrease in the richer's "welfare," for amounts which are equal (concavity of the functions f in an additive evaluation $\Sigma f(x_i)$).

4.1.2 "Social welfare"

If the overall evaluation of the distribution is both separable-independent and symmetrical, the ordinal function W has specifications of the form $\Sigma f(x_i)$. This cannot describe classical utilitarianism $\Sigma u_i(x_i)$ because the same function f applies to all x_i.[10] If this refers to "welfare," this is a concept different from the individuals' evaluations of their own welfare. This raises two questions: what can this evaluation mean, and what can it be? The second question is in part eschewed by the consideration of comparisons that holds for *all* functions f that are increasing (benevolence) and concave. This latter property means that an extra euro increases evaluation or "welfare" more the lower the income to which it is added. It also is a property of "satiation" in the evaluation or "welfare" effect of individual income.

4.1.3 Concentration curve and Lorenz curve dominances

Denote as y_m the sum of the m lowest x_i. That is, if the numbering i of the x_i are rearranged in such a way that the new x_i are in a non-decreasing order ($x_1 \leq x_2 \leq \ldots \leq x_n$, that is, $i>j$ implies $x_i \geq x_j$), y_m is $y_m = \sum_{i=1}^{m} x_i$. Then, $y_n = \Sigma x_i = X$, the total amount.

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[10] However, this additive form is the case where the remark of note 9 applies.

Elementary textbooks of statistics call the curve of the y_m as function of m (or of m/n) the *concentration curve* of the distribution of the x_i.

The Lorenz curve of this distribution is y_m/X as a function of m/n.

When the x_i are all equal, these two curves are straight lines with these x_i and 1 as respective slopes.

A curve is said to be above another when it is somewhere above and nowhere below.

A distribution *concentration-dominates* another when its concentration curve is above that of the other, that is, for distributions $(x_1,\dots x_n)$ and $(x'_1,\dots x'_n)$, $y_m \geq y'_m$ for all m and $y_m > y'_m$ for at least one m. *Lorenz-domination* is similarly defined for Lorenz curves. Both comparisons coincide when comparing distributions with the same total $X = X'$, that is, in "constant-sum comparisons." Then, a preference for a higher concentration or Lorenz curve is called isophily (*isophilia* is the Greek term for inequality-aversion).

4.1.4 Averaging

A distribution is in a sense less dispersed than another if all its items are averages of those of the other. Distribution $(x'_1,\dots x'_n)$ is a (linear convex) average of distribution $(x_1,\dots x_n)$ when $x'_i = \Sigma_j\, a_{ij}\, x_j$ with $a_{ij} \geq 0$ for all i and j and $\Sigma_j\, a_{ij}=1$ for all i. If the total sums are equal $X = X'$, notably for a redistribution, this implies the last equality of

$$\Sigma\, x'_i = \Sigma_{i,j}\, a_{ij}\, x_j = \Sigma_j (\Sigma_i\, a_{ij})\, x_j = \Sigma_j\, x_j,$$

and therefore

$$\Sigma_j\, (1 - \Sigma_i\, a_{ij})\, x_j = 0.$$

We consider such transformations that are independent of the initial distribution (the a_{ij} do not depend on the x_k), and applicable to all distributions. The foregoing identity then implies $\Sigma_i\, a_{ij}=1$ for all j. Such a_{ij} constitute a *bistochastic matrix*, i.e., a non-negative matrix whose sums of the elements in each row and in each column amount to 1. This transformation of x into x' is an *averaging*.

If a_{ii}=1 for all i (hence a_{ij}=0 if $i \neq j$), $x'_i = x_i$ for all i, nothing is changed, the transformation is an identity. If all a_{ij} are only zero or one, the transformation is a permutation of the x_i. If $a_{ij}=1/n$ for all i, j, $x'_i = \bar{x}$ for all i (a "complete averaging"). If, for $0 \leq \alpha \leq 1$, $a_{ij}=\alpha/n$ for all i, j with $i \neq j$ and $a_{ii}=1-\alpha+\alpha/n$ for all i, $x'_i = (1-\alpha)x_i + \alpha\bar{x} = x_i + \alpha\cdot(\bar{x} - x_i)$ for all i. This is a *concentration* of the x_i (a uniform linear concentration towards the mean): each x'_i is an average between x_i and the mean $\bar{x}$, it goes the same fraction α of the way towards the mean; the concentration amounts to an equal redistribution of the same fraction α of the x_i; it amounts to a decrease of all incomes in the same proportion followed by an increase of the same amount (which restores the total amount). A progressive transfer is a particular averaging: if $x_i > x_j$, $0<t<1$, $a_{ii}=a_{jj}=1-t$, $a_{ij}=a_{ji}=t$, $a_{kk}=1$ for all $k \neq i, j$, and $a_{kl}=0$ for the other entries, $x'_i = x_i - t\cdot(x_i - x_j)$, $x'_j = x_j + t\cdot(x_i - x_j)$, and $x'_k = x_k$ for all $k \neq i, j$. Of course if all the x_i are equal, all the x'_i are also equal to them. Moreover, an averaging of an averaging is an averaging.

4.1.5 Share reshuffling

Divide each individual income into a series of shares, each share being the same fraction of the income for all incomes. Then, reshuffle the shares corresponding to the same proportion among the individuals, that is, perform a permutation of these shares among them. The permutations of the shares for the various fractions are unrelated. Formally, consider numbers $\lambda_k > 0$ with $\Sigma\lambda_k = 1$, and permute the shares of each k, $\lambda_k x_i$, among the individuals i, with independent permutations.

4.1.6 Mixtures

Denote as $x=\{x_i\}$ the vector of the incomes x_i. A permuted vector of x is x^π obtained by permuting the x_i of x by the n-permutation π (i.e., $x_i^\pi = x_{\pi(i)}$ for all i). The absence of relevant individual characteristics other than their incomes x_i implies that the x^π are equivalent. Then, a *mixture* of a distribution x is an average (a linear convex combination) of the x^π, $x' = \Sigma\lambda_\pi x^\pi$ with $\lambda_\pi \geq 0$ for all π and $\Sigma\lambda_\pi = 1$.

Since, in share reshuffling, if one writes π_k the permutation corresponding to share k, the result is $x' = \Sigma\lambda_k\ x^{\pi_k}$, mixtures and share reshuffling are clearly equivalent (each instance of one in an instance of the other). These transformations are not permutations when $\lambda_\pi \neq 1$ for all π for a mixture, and, for a share reshuffling, $\lambda_k \neq 1$ for all k (i.e., there are at least two shares) and the permutations of the shares are not all identical.

A transformation that is not, in fact, a permutation is called *strict*.

4.2 The fundamental equivalences of ethical inequality comparisons

Each of these properties has a flavour of comparing more or less unequal distributions. Their meaning in this respect is very strongly reinforced by the fact that they are mathematically equivalent.

Indeed, when comparing 2 distributions $x=(x_1,\ldots x_n)$ and $x'=(x'_1,\ldots x'_n)$ with the same amount $X = X'$, the following properties are equivalent.

1) x' can be obtained from x by a sequence of progressive transfers.
2) The concentration or Lorenz curve of x' is above that of x.
3) $\Sigma f(x') > \Sigma f(x)$ for all increasing and strictly concave functions f.
4) x' is a strict averaging of x.
5) x' results from a strict share reshuffling of x.
6) x' is a strict mixture of x.

Moreover, if the distributions can have different amounts, say $X' \geq X$, the following properties are equivalent.

1) X' can be obtained from X by a sequence of progressive transfers or increases in incomes.
2) $\Sigma f(x'_i) > \Sigma f(x_i)$ for all increasing strictly concave functions f.
3) The concentration curve of distribution x' is above that of distribution x.

Clearly, these relations cannot be both ways between two distributions; if they hold from x to x' and from x' to x'', they hold from x to x'' (transitivity). They thus constitute an ordering of the distributions. For distributions with the same total amount, this is an important sense of comparisons by more or less unequal. Yet, they do not compare all distributions: they do not compare them when their concentration curves intersect. Other criteria can then be added.

An evaluation function $W(x)$, increasing, symmetrical and such that $W(x')>W(x)$ when x' relates to x as in the preceding relations, and the corresponding ethical evaluation-consistent inequality indexes, are called *rectifiant*, or, respectively, Schur-concave and Schur-convex[11] (the functions $\Sigma f(x_i)$ with increasing and concave f constitute a sub-class of such functions).

Finally, there are types of redistributions or transformations of distributions that are more inequality-reducing structures than the others. The two polar cases of the particularly inequality-reducing transformations are the *concentrations* in which all incomes diminish their distance to the mean in the same proportion, and *truncations* where all incomes above a level are reduced to this level and all below a lower level are augmented to this level. Both have important applications in normative economics – this is notably the case for concentrations in the theory of optimum distribution, taxation and aid.

5. Inequality under co-variations of incomes

The foregoing mainly emphasizes the effects of transfers or redistributions on inequality, hence comparisons of the inequality of distributions with the same total amount. However, cases in which all incomes vary in the same direction are also important. Does general growth, or an equal distribution of a benefit or a charge, augment or diminish inequality? This depends on the relevant concept of inequality.

The two polar cases are those in which inequality does not change when all incomes vary in the same proportion or by the same amount. In the former case, inequality is what the sciences call an *intensive* property. In the latter case, inequality is said to be *equal-invariant*.

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[11] After I. Schur whose articles of 1922, 1923 and 1936 first considered the effects of the transfer principle and averaging on such functions (rectifiance means, more generally, the satisfaction of the transfer principle whether the functions are symmetrical or not).

Measures of inequality derived from a separable evaluation that are *intensive* are the *relative* inequality with a *power* or a *logarithmic* individual welfare function ($f(x_i) = x_i^{\alpha}$ with $\alpha>0$, or $\log x_i$), and those that are *equal-invariant* are the *absolute* inequality with an *exponential* individual welfare function ($f(x_i)=1-e^{-\beta x_i}$, $\beta>0$). There results, in particular, that one cannot derive both an intensive and an equal-invariant measure of inequality from the same separable ethical evaluation.

Nevertheless, there is another class of measures of inequality, the *synthetic* measures, with an absolute form $I^a(x)$ and a relative form $I^r(x) = I^a(x)/\bar{x}$, such that the relative form is intensive and the absolute form is equal-invariant. There results that the absolute form is also “extensive,” that is, multiplied by a scalar when all incomes are. These absolute forms are the linearly homogeneous functions of the differences ($x_i - \bar{x}$) or ($x_i - x_j$). They include some of the most common measures of inequality such as the Gini index $\Sigma|x_i - x_j|$, $\Sigma|x_i - \bar{x}|$, or the standard deviation.

Moreover, one can derive, from a separable ethical evaluation, measures of inequality that are intermediate between the intensive and the equal-invariant measures. The simplest case is the “income-augmented” intensive measures, which apply the intensive measures to new variables that are the incomes plus a non-negative constant. The measures are intensive when the constant is zero and equal-invariant when it tends to infinity.

For intensive or equal-invariant measures, one can reduce the comparison of the inequality of two distributions to constant-sum comparisons by respectively multiplying or increasing all the incomes of one of the distributions by the same number.

6. Conclusion

The foregoing properties constitute only the rock-bottom of the standard economic theory of unjust inequality. Many other properties are added. In particular, they describe the effects, on this inequality, of: transfers depending on the levels of or differences in incomes; the addition of several types of incomes to the same people; the aggregation of populations with intra-group and inter-group inequalities; growth; the income tax; characteristics which may relevantly differentiate the persons such as needs, size and type of family, labour provided, merit or desert, or various rights; judgments that violate the transfer principle, for instance because they attach importance to clusters of incomes (size of income classes); and so on.

The theory then considers the inequalities in other items than income or a single quantity, notably the multidimensional inequalities in a bundle of goods (to begin with in both income and labour or leisure, or in income, health, education and housing); inequalities in various types of freedom, power or opportunities; inequalities in ranks or status; etc.

The nature of the items often implies particular properties of the comparison and measures of inequality. This happens even with the simplest case of quantities. For instance, if health is measured by the duration of life, it may be, on average, better to die at 35 rather than at 34 than to die not only at 95 rather than at 94 (concavity of the function *f*), but also at 5 rather than at 4 (non-concavity of *f*).

Other judgments about distributions and their comparisons, with important ethical dimensions, are closely related to those about inequality. In particular, poverty is not only related to the issue of inequality: the economic theories of both questions are very close to one another, almost identical.

In other cases, the basic reference is not equality but some other particular distribution. It can, for instance, be the outcome of markets, which has a possible moral justification from freedom of exchange (or self-ownership). Then, the relevant concept is the degree of equalization achieved by redistributions from this state. For example, present-day redistributions at national levels are equivalent, in this respect, to fully equalizing the incomes of 1 to 2 days per week. Such durations turn out to be richly meaningful measures of the degree of equalization or solidarity in the community.

Still other structures of inequality have important social-ethical consequences or implications. Some refer to the clustering of incomes and the social situations they represent. Lower inequality a priori refers to a unique cluster around the mean. But the existence of several clusters can also manifest a kind of lower inequality, notably when the effects of two incomes becoming closer to each other by some amount are more important the closer they are.[12] However, clusters have other kinds of social effects (with possible ethical consequences) than only distributive injustice. They manifest social segmentation when there are more than one. This can be related to a social structure of classes, castes, or other types of social hierarchies. A small cluster of incomes at the bottom or at the top of the distribution manifests isolation or exclusion of the poor or "elitism" of the rich.[13] There is more polarization in society when there are two clusters which are more apart and tend to be of more equal size.[14]

Economics has accompanied these ethico-logical analyses with very numerous empirical measures and comparisons of inequality, in income and other items. In practically all cases, ethical judgments are present, at least implicitly. In all cases, however, other criteria or measures of inequality give different results, and the discussion of this issue is often absent and almost always very thin and feeble. Thus, there happened to be an urgent need for deepening the ethical aspect of the structure, comparisons and measurement of inequalities. This was done by the analysis of the moral judgments, sentiments, and emotions aroused by inequalities of various items and structures in various contexts. Several methods are jointly used for this purpose: elaborate questionnaires and statistical and semantic analyses of the answers; phenomenological and conceptual analysis; discourse analysis of various texts; and comparison with the other judgments, sentiments and emotions elicited by inequalities, such as compassion, pity, envy, or sentiments of inferiority or superiority. The advanced study of economic inequality is more and more based on the elaborate analysis of the psychology of moral and social sentiments, which in turn it importantly stimulates.

References and a short bibliography

The literature on economic inequality is very large and cannot be presented here. A source of references is the handbook edited by J. Silber. Among the other volumes devoted to this topic, the textbook by P. Lambert is probably the most complete.

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[12] For instance for inequality indexes of the form $[\sum_{i,j} (*x_i ! x_j*)^{\alpha}]^{1/\alpha}$ with $\alpha<1$.

[13] G. Field (1993).

[14] J. Esteban and D. Ray (1994).

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