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On real economic freedom

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Abstract Among the various concepts of freedom important for economics, ranking or measuring the freedom of choice provided by budget sets has an important place. The volume ranking has strange properties and cannot be justified by unit invariance and symmetry. The pointed distance (of the budget hyperplane from the origin along some line) provides a measure or ranking that coincides with the standard "purchasing power" or real income. The linear price index is practically unavoidable for measuring or ranking freedom. This is applied to the determination of income distribution and taxation implied by the equal freedom of choice of different domains. Concepts of equal or compared potential freedoms and utility-freedom relate freedom analysis to the basic classical concepts of fair allocation (equity-no-envy, egalitarian equivalence, etc.). The crucial difference between the two opposite concepts of invariance in comparisons is emphasized.

1 Liberties

The three types of freedom are relevant for economics.[1] The expressions "economic freedom", "a free economy", perhaps "free enterprise" refer to *freedom from forceful interference*, designated by various names such as "negative freedom" (Kant, John Stuart Mill, Berlin), "civic freedom" (John Stuart Mill), "social liberty", "process freedom", the "basic rights" or "basic liberties" when emphasis is put on specific domains, and "formal freedom" (Marx). This liberty implies free exchange, the aspect

[1] For a definition of free (that is, caused by the will or reason—which needs the will to implement it—the will and reason being particular mental processes), the distinction of various relevant and important characteristics and types of freedom, and the reasons to value it, see Kolm (1982, 1996a, Chap. 2, 1996b, 1998). Berlin (1958) notes that historians of ideas have counted about 200 conceptions of freedom.

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concerning the economy and economics.[2] When the other means (assets, rights, etc.) of an agent are added, one obtains her domain of possible choice which offers "real freedom" (Marx). The third kind of freedom is Rousseau–Kant's "autonomy", i.e., the choice of one's own criterion of choice.[3]

The first two freedoms only are standard in economics. Issues of having, or of there being, more or less freedom, or equal freedom, are raised, in particular for normative reasons. The first type, "social liberty", can be held at satiety for all (when several persons' intended acts are incompatible, the conflict is attributed to the other means they use, such as various possible property rights). Hence this liberty can be equal for all in this sense. The second type of liberty leads to comparing freedoms of choice. Economics' emphasis on free exchange (and competition) gives a particular importance to the comparison of the liberty provided by various budget sets.

2 Budget sets and freedom of choice[4]

Let us consider n goods indexed by i. The quantity of good i writes $x_i \in \Re_+$, $x = \{x_i\}$ denotes the vector of the x_i, $p_i \in \Re_{++}$ the price of good i, $p = \{p_i\}$ the vector of prices, $px = \Sigma p_i x_i$ the scalar product, and $y \in \Re_+$ is income. Given y and p, a *budget set* is

$$B(y, p) = \left\{x \in \Re^n_+ : px \leq y\right\}.$$

The corresponding *budget hyperplane* is

$$\beta(y, p) = \left\{x \in \Re^n_+ : px = y\right\}.$$

We also denote $a_i = y/p_i$ the ith coordinate of the intersect of $\beta(y, p)$ with the ith axis, $a = \{a_i\}$ the set of the a_i; and $q_i = p_i/y = 1/a_i$ the *income-normalized* or *normalized* price of good i, with $q = \{q_i\}$ denoting the vector of the q_i. We have

$$B(y, p) = B(1, q)$$

and

$$\beta(y, p) = \beta(1, q)$$

[2] A priori free exchange without the forceful interference of a third party except if this results from another free agreement (possibly a collective political one). This social liberty also implies rights on the proceeds of the free actions. Forceful interference includes aggression, theft, etc., and also steady social domination the absence of which is the political "republican freedom".

[3] Note that what Berlin calls "positive freedom" is not the freedom offered by the other means, as it has been mistakenly understood and presented by some economists, but, rather, something like republican freedom transformed by a division of the self and a kind of autonomy into either mastering one's desires or submitting oneself to the dangerous dictature of some ideal (the implicit but unambiguous intention is to contrast totalitarian ideologies and classical liberalism). Mental liberty is the central issue of Kolm (1982).

[4] Most results of the present study are drawn from Kolm (2004a,b).

Let $\tilde{B} = \{B(y, p)\}$ denote the set of budget sets for all possible (y, p).

The issue is some agent's choice of one $x \in D$ in a *domain of choice* $D \subset \Re^n_+$, and the corresponding "freedom" of this choice offered by D. Various questions, noted or discussed shortly, give an interest in ranking these D according as to their providing more or less freedom of choice, and sometimes in providing a measure of this freedom. Ranking suffices in particular to define domains providing equal freedom, which turns out to have a very important application for the optimum or fair distribution and taxation of income.

We will denote, for $x, x' \in \Re$, $x \geqq x'$ as $x_i \geq x'_i$ for all i; and $x \geq x'$ as $x \geqq x'$ and $x \neq x'$.

Free disposal for good i implies that, if x is possible, x' with $x'_j = x_j$ for all $j \neq i$ and $x'_i < x_i$ is possible. Budget sets $B(y, p)$ are consistent with this property.

If $\succsim$ denotes a weak *freedom ordering* of the domains $D \subset \Re^n_+$ (with the corresponding $\succ$ and $\sim$), one has $D = D' \Rightarrow D \sim D'$. The classical question whether $D \supset D'$ implies $D \succ D'$ or $D \succsim D'$ is normally not raised for $D, D' \in \tilde{B}$. Indeed, this question refers to the case in which, for all $x \in D/D'$ there is $x' \in D'$ which is preferred (or equivalent) to x for the relevant choices in these domains. In particular, if, for each good, either there is free disposal or a larger quantity is always preferred, this would happen with $x' \geq x$. However, $B(y, p) \supset B(y', p')$ implies that $x \in B(y, p)/B(y', p') \Rightarrow \nexists x' \in B(y', p') : x' \geqq x$.

Hence, the useful assumption for the *inclusion property* of the freedom ordering of budget sets is prima facie

$$B(y, p) \supset B(y', p') \Rightarrow B(y, p) \succ B(y', p').$$

For $x, x' \in \Re^n$, x is said to *dominate* x' if $x \geq x'$.

For $D \subset \Re^n_+$, its *disposable extension* is

$$\overline{\overline{D}} = \{x \in \Re^n_+ : (\exists x' \in D : x' \geqq x)\}$$

and its *undominated set* is

$$\overline{D} = \{x \in D : (\nexists x' \in D : x' \geq x)\}.$$

Then, $\overline{D} \subseteq D \subseteq \overline{\overline{D}}$. A domain D is said to be *disposable* when $D = \overline{\overline{D}}$. Let C denote the set of compact sets of $\Re^n_+$. Then $D \in C \Rightarrow \overline{D} \neq \emptyset$. D and $\overline{\overline{D}}$ have the same undominated set, and D and $\overline{D}$ have the same disposable extension.

There are two possible reasons to focus on the undominated sets, *free disposal* and *preference for larger quantities*. It suffices that, for each good, any one of these two properties holds.

Free disposal for all goods implies that, if x is possible, any $x' \leq x$ also is. Then, for domains of possibility, $D = \overline{\overline{D}}$: all are disposable. If, moreover, $D \in C$, $(\overline{\overline{\overline{D}}}) = D = \overline{\overline{D}}$ and $\overline{D}$ is the smallest subset of D the possibility of which implies that of D. Then, $D \sim \overline{D}$ and, to rank or measure freedom of choice, D can be replaced by $\overline{D}$.

Alternatively, any choice preferring larger amounts of goods chooses a $x \in \overline{D}$ in D. Such a condition for any good (or all) introduces a consideration of preferences or "utility", but certainly minimally.

For budget sets, $\overline{B}(y, p) = \beta(y, p)$, and $\overline{\overline{B}}(y, p) = \overline{\overline{\beta}}(y, p) = B(y, p)$. Budget sets are disposable.

Finally, we will consider a representation of the freedom ordering by ordinal functions $F(D)$ with $F(D) \geq F(D') \Leftrightarrow D \succsim D'$. For $D \in \tilde{B}$, we will write

$$F(y, p) = F(1, q) = G(q) = H(a).$$

3 Freedom as volume

Ranking or measuring the freedom of choice offered by domains $D \subset \Re^n_+$ by their volume may permit to take account of the contribution of each part of D. This is discussed, notably for budget sets $D \in \tilde{B}$, in Kolm (2004a).[5] Xu (2004) provides, for budget sets, an axiomatic justification considered in Sect. 4. This volume ranking or measure of freedom of choice raises a priori a number of problems. Some of them do not occur when $D \in \tilde{B}$, such as those raised by "thin" parts of D of dimension lower than n which have no volume in $\Re^n$. In particular, $x_i = 0$ for any good i in all of D gives a volume of zero, lower than that of other domains possibly very restricted. Moreover, the volume of a budget set tends to zero when any price p_i becomes very high for given y and p_j for $j \neq i$, no matter how unimportant the corresponding good is (a high price, however, implies that some buyers desire the good).

At any rate, a notable problem is that, with free disposal, the possibility to choose $x' \leq x$ adds nothing to that of choosing x (the same result holds if it is known that more of any good is better, but free disposal suffices). This would lead one to make the ranking or measure depend on the undominated sets $\overline{D}$ only. However, the dimension of this set is lower than n and its volume is zero. One could then use the volumes of $\overline{D}$ in their space, often of dimension $n-1$. However, this ranking differs from the ranking of the volume of D. For $D \in \tilde{B}$, in particular, the ranking of the volumes of $B(y, p)$ differs from the ranking of the $(n-1)$-volumes of $\overline{B}(y, p) = \beta(y, p)$. For example, for $n = 2$, a constant volume of B gives budget lines tangent to the same equilateral hyperbola, and their segments between the axes, $\overline{B}$, have lengths that extend to infinity (for very low or high p_i/p_j); whereas such segments of equal length of 1 correspond to volumes of B from 1/4 to zero.

Moreover, attributing the same contribution to freedom to parts of D with equal volume also raises questions.

Finally, the volume is an increasing function of

$$(\Pi a_i)^{1/n} = y/(\Pi p_i)^{1/n}. \tag{1}$$

[5] Pages 423–425. In 1972, Jean-Marc Oury proposed to study volume as freedom of choice as his doctorate dissertation but was discouraged by the present writer.

This is a "real income" having, as price index (homogeneous of degree one in the set of prices), the harmonic mean of prices.[6]

This price index and this ranking imply fascinating properties. The volume tends to zero, and hence freedom of choice becomes worse than in any other situation, when the price of any good becomes very large—for example the price of ivory cuff links—for given other prices and y. If the price of any one good is multiplied by the same number, the resulting freedom of choice is the same whatever this good (for instance, one may multiply by 10,000 the price of water or that of ivory cuff links). Freedom of choice is not changed if prices of goods are permuted (for example those of cars and of sandwiches; the crazy store manager who allocates randomly the price labels on the items does not affect the customers' freedom of choice).[7]

Now, the changes considered by Xu are changes in units for measuring the quantities of goods. He wants these changes to have no influence on the ranking. However, they entail the corresponding inverse changes in prices. Then, if b_i denotes any quantity of good i, what is unit invariant are products $b_i p_i$. And the standard price indexes are, indeed, of the form $i = \Sigma b_i p_i$, which define so-called "real incomes" or "purchasing powers" $y/\Sigma b_i p_i$. Such expressions have weighed prices, the weights have the magnitude of quantities of the corresponding goods, and the expression is neutral in both the numéraire and the units of goods; whereas expression (1) is symmetrical in prices, neutral in numéraire, but not neutral in units of goods.[8]

4 Contravariance and symmetry

However, Xu proposes an axiomatic justification of the volume ordering of the freedom of budget sets. He states three axioms. (1) The inclusion property: a superset of possibilities offers more freedom. (2) "Invariance to scaling effects" which is supposed to represent the unit invariance property: the ranking should be invariant under changes of units of the goods. (3) A symmetry property stating that permutations of the goods do not change the ranking. The idea that a representation of freedom should not depend on preferences guiding choice is used to justify both the symmetry property and the property of the obtained volume ranking that pieces of the possibility set with the same volume have the same weight. It is suggested that this latter property corresponds to counting the items in the case of a finite set of alternatives, but, as a fact, this seems questionable.

$B(y, p) = B(1, p/y)$ is fully represented by the set a of the $a_i = y/p_i$. Xu uses this description. Both axioms 2 and 3 raise problems. The symmetry axiom is applied as an invariance of the ordering to permutations in a_i. This amounts to the invariance to permutations in prices just noted since $a_i = y/p_i$. Moreover, unit invariance is applied as the ordering not being changed by the replacement of any a_i by λa_i with

[6] Kolm (2004a).

[7] Kolm (2004a).

[8] Savaglio and Vannucci (2009) point out that some restrictions in application permit avoiding some shortcomings of the volume ranking. Each good should be necessary, which avoids $x_i \equiv 0$ (but this issue does not arise for budget sets anyway). Excluding free disposal and considering unknown preferences which may be non-monotonic and may choose any bundle of goods leads to excluding the restriction of D to $\overline{D}$.

any $\lambda > 0$. Then, indeed, representing the ordering by the function $H(a)$, if for any a, a', i, and any $\lambda > 0$,

$$H(a) \geq H(a') \Rightarrow H(a_1, \ldots, \lambda a_i, \ldots, a_n) \geq H(a'_1, \ldots, \lambda a'_i, \ldots, a'_n), \quad (2)$$

which implies

$$H(a) = H(a') \Rightarrow H(a_1, \ldots, \lambda a_i, \ldots, a_n) = H(a'_1, \ldots, \lambda a'_i, \ldots, a'_n), \quad (3)$$

then function H is of the form

$$H(a) = h(\Pi a_i^{\alpha_i}) \quad (4)$$

for some increasing function h and $\alpha_i > 0$ for all i.[9] Then symmetry (axiom 3) is applied to function $H(a)$ and therefore implies $\alpha_i = \alpha$, the same for all i. The result is ordinally equivalent to Πa_i, and therefore to the volume ranking.

An elementary proof of this result and of Xu's theorem consists in noticing that it reduces to a standard property of linear manifolds by the change of varibles $\tilde{a}_i = Log\, a_i$ and, for a transformation of a_i into $\lambda_i a_i$, $\tilde{\lambda}_i = Log\, \lambda_i$ (see section 11).

However, the ranking represented by inequalities $H(a) \geq H(a')$ is a comparison of *actual* properties of budget sets. These properties, being actual, are independent of the units of the goods, which are arbitrarily chosen to measure their quantities. Therefore, if function $H(a)$ is used to compare these properties, it should be independent of the units of the goods. That is, if units of the goods change, function $H(a)$ correspondingly changes so that its value does not change (it remains just a given specification of an ordinal function). As a consequence, the ordering represented by the comparison of these values does not change either. This invariance of $H(a)$ in the units of measurement amounts to $H(a) = \hat{H}(\{a_i/b_i\})$ where $b_i > 0$ has the magnitude of a quantity of good i. Hence, when the unit of good i becomes λ_i times smaller, the measure a_i is multiplied by λ_i, but b_i is too, and therefore the value $\hat{H} = H$ does not change. Therefore, comparisons $H(a) \geq H(a')$ do not change either. Hence, neutrality of the comparison under this change of units is not described by relations (2) and (3). Therefore, it does not imply the consequences derived from these relations, such as the volume ranking. This change in function to erase the effects of the arbitrary choice of units is the classical contravariant transformation basic in all sciences.[10] What is

[9] More explicitly, let vector a take four values a^1, a^2, a^3, a^4, such that $a_i^3 = \lambda a_i^1$, $a_i^4 = \lambda a_i^2$, and, for all $j \neq i$, $a_j^3 = a_j^1$ and $a_j^4 = a_j^2$. Then $H(a^1) \geq H(a^2) \Longrightarrow H(a^3) \geq H(a^4)$ for any a^1, a^2, i, and $\lambda > 0$ if and only if form (4) holds.

[10] Xu works with orderings, and hence his orderings can incur the contravariant transformation. His orderings are representable by ordinal freedom functions. Actually, the elegance of the derivations that can be obtained by omitting the logical requirement of contravariance is often so appealing that a number of famous studies in normative economics could not resist this fallacy. Another case has exactly the same structure. Suppose one wants to justify Nash bargaining solution. Then, take H to be a social welfare function, and $a_i = u_i(x) - u_i(x_0)$ where u_i is a cardinal utility function, x the social state, and x_0 a particular reference state. Then a_i is defined up to an arbitrary multiplicative factor independently for each i. If this is interpreted as implying condition (2), this leads to form (4). An appeal to symmetry may then produce Πa_i

"scale-invariant" is function H and not only comparisons $H(a) \geq H(a')$. Of course, the former implies the latter. Actually, the compared a and a' need not even be measured with the same units of the goods (we should be able to compare budget sets of an Englishman and of a Frenchman even if some goods have quantities measured by their weight or their surface). Since $a_i/b_i = y/b_i p_i = 1/b_i q_i, \hat{H}(\{a_i/b_i\}) = \hat{G}(\{b_i q_i\})$. A change in p_i or q_i due to a change in units of good i jointly changes b_i and $b_i p_i$ and $b_i q_i$ are invariant.

The volume ranking is still formally a possibility: it occurs when function $\hat{H}$ is a product of its arguments. But this is no longer the only and necessary possibility. It would require other justifications. Note that a contrasting other possibility is that function $\hat{G}$ is a sum of its arguments (for an inverse ranking). This ranking is then defined by the function $\Sigma b_i q_i = \Sigma b_i p_i / y$, and hence the direct ranking is provided by the inverse $y/\Sigma b_i p_i$ which is the classical purchasing power or real income.

Note also that the interpretation of unit neutrality by Xu's "scale invariance" also has the interesting consequence that all utility functions are Cobb–Douglas. Consider, indeed, a_i as a quantity of good i consumed by an individual and H as the utility function. Individuals are a priori indifferent as to whether bread is measured in kilograms or in grams. An interpretation by condition (2) yields form (4) ordinally equivalent to a Cobb-Douglas function.

5 The philosophy of Xu's paradox: economics, as physics, is not mathematics, and a generalization

Xu's paradox—deriving a hardly acceptable conclusion (the volume ranking) from a necessary condition (unit invariance)—however, reveals an interesting philosophical point. Two facts are formally identical but semantically different: unit invariance and scale invariance. The former is a tangible property, a property of the "real world", here economics but it could be physics, mechanics, engineering, chemistry or biol-

Footnote 10 continued
as a specification of the social welfare function (Nash's solution for $n = 2$). Yet, this appeal to symmetry has no justification, and the appeal to condition (2) bypasses the contravariant transformation. In another story, H is again a social welfare function, a_i is a cardinal utility of individual i, and H is also required to be cardinal (for instance, they are the corresponding von Neumann–Morgenstern specifications). Then, each of these functions being defined up to an increasing affine function, plus an appeal to symmetry, would require that H is a utilitarian sum. Yet, this omits the contravariant transformation, and the symmetry has no justification. Another example is provided by Maskin (1978) "proof" of utilitarianism (see Kolm 1996a, Chap. 11). There are two a priori assumptions: individuals' utilities are co-cardinal, of the form $\alpha u_i + \beta$ for individual i with arbitrary $\alpha > 0$ and β, and the social maximand is additive and symmetrical, $\Sigma f(u_i)$. Then, if the maximization remains the same if the u_i are jointly transformed into such $\alpha u_i + \beta$, function f has to be affine and the maximand amounts to Σu_i. However, this indeterminacy in the definition of the u_i means, rather, if function f is meaningful, that so transforming the u_i contravariantly transforms it into function φ such that $\varphi(\alpha u_i + \beta) = f(u_i)$. A similar fallacy underlies the argument, which has been proposed, that an index of income inequality should be "scale-invariant" or homogeneous of degree zero because it should not change if the unit of measure of incomes changes (measure in dollars, cents or pounds). Yet, an other index need not be unit-neutral and can incur the contravariant transformation when the unit for measuring incomes change. Note that the very term "scale-invariant" may suggest the mistake (homogeneity of degree zero as a real property, that is dependence on ratios—here of incomes—only, characterizes the measures that engineers and physicists call "intensive").

ogy. All of science faces it by representing the world by unit neutral functions, which means that, when units change, the function incurs the corresponding contravariant transformation. Then, an ordering represented by such a function is ipso facto unit invariant. In contrast, scale invariance is a mathematical property. One may thus ask which orderings are scale invariant (given that the representative functions are not correspondingly unit adjusted). This is formally interesting, and some related properties may have applications to actual problems. Then Xu's result is a particular case of a more general theorem which covers the cases in which changes relative to each good are independent from one another (Sect. 14): *the ordering does not change when the variables* q_i *are transformed into* $q_i + \varepsilon_i g_i(q_i)$ where g_i *is a* $\Re \to \Re$ *function and* $\varepsilon_i \to 0$, *if and only if the representative function G(q) is additively separable, i.e., it can be written as* $\Sigma f_i(q_i)$ *with* $\Re \to \Re$ *functions* f_i, *with* $g_i f_i' = c_i$, *a constant.*

In a particular case, $g_i = s_i q_i$ where s_i is a constant, q_i is multiplied by $1 + \varepsilon_i s_i$ and this is a scale invariance. Then $f_i = \beta_i$ Log q_i, where β_i is a constant, and if this holds for all i, G is ordinally equivalent to $\Sigma\beta_i$ Log q_i, hence to $\Pi q_i^{\beta_i}$, and therefore, with symmetry implying $\beta_i = \beta$ for all i, inversely to Πa_i and to the volume. Another particular case is $g_i = t_i$, a constant. Then $f_i = \rho_i q_i + \sigma_i$ with constant ρ_i and σ_i, and if this holds for all i, G is ordinally equivalent to $\Sigma \rho_i q_i$. This is a case studied by Miyagishima (2009). Contrary to the volume ranking, however, this case happens to be highly significant on real grounds since this function is simply the inverse of $y_i/\Sigma\rho_i p_i$ which is the classical "purchasing power" or "real income" for income y, measured with a standard linear price index with weights ρ_i (the reference quantity of good i), and which corresponds to the ranking of budget sets by the "pointed distance" (Kolm 2004a,b, 2008).

6 The pointed distance

Definition For any vector $b = \{b_i\} \in \Re^n_+/\{0\}$ and any domain $D \subset \Re^n_+$, the *b-maximum* of D is defined as

$$M(D, b) = \sup \lambda \in \Re : \lambda b \in D.$$

Hence, if $D \in C$,

$$M(D, b) = \max \lambda \in \Re : \lambda b \in D.$$

Then, $M(D, b)$ is the highest number of bundles of goods b obtainable in D, with decimals corresponding to the same fraction of the b_i. For $D \in \tilde{B}$, $M(D, b)$ is also the *b-pointed distance* of the budget hyperplane:

$$P(y, p, b) = M\left[B\left(y, p\right), b\right] = M[\beta(y, p), b] = \lambda \in \Re : \lambda b \in \beta\left(y, p\right)] \quad (5)$$

This is the distance from the origin of the intersect of the budget hyperplane $\beta(y, p)$ with the line $l(b)$ from the origin bearing vector b, measured with the length of b as unit (Fig. 1).

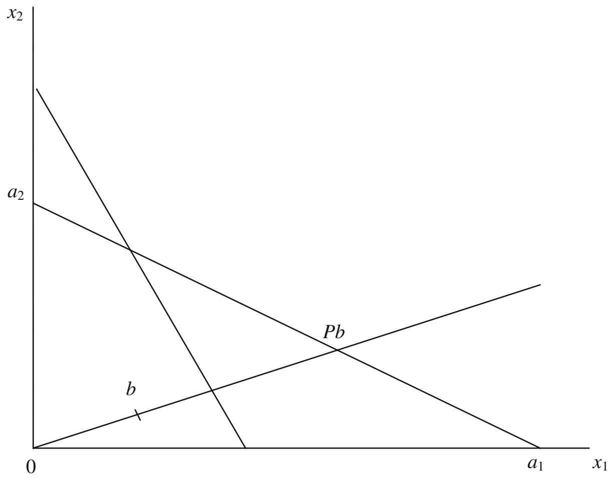

Fig. 1 The pointed distance

Note that, in this respect also, comparing budget sets does not have a shortcoming present with more general domains in the case of inclusion. Indeed, for D, $D' \subset \Re^n_+$,

$$D \supset D' \quad \Rightarrow M(D, b) \geq M(D', b) \quad \text{for all } b \in \Re^n_+/\{0\},$$

but this may be $M(D, b) = M(D', b)$ for some b. However, if domain D' *is disposable*, then

$$D \supset D' = \overline{\overline{D'}} \quad \Rightarrow \exists b \in \Re^n_+/\{0\} : M(D, b) > M(D', b).$$

Indeed, take $b = x \in D/D'$. Then $M(D, x) \geq 1$. Moreover,

$$M(D', x) \geq 1 \Rightarrow \exists x' \in D' : x' \geqq x.$$

But then $D' = \overline{\overline{D'}} \quad \Rightarrow x \in D'$, contrary to the definition of x. Hence $M(D', x) < 1$ and $M(D, x) > M(D', x)$.

For D, $D' \in \tilde{B}$, in contrast,

$$B(y, p) \supset B(y', p') \Rightarrow P(y, p, b) > P(y', p', b)$$

for all $b \in \Re^n_{++}$, and for all $b \in \Re_+/\{0\}$ if

$$\beta(y, p) \cap \beta(y', p') = \emptyset.$$

7 The standard "purchasing power"

When $D \in \tilde{B}$, the freedom of choice is provided by the act of *purchasing* (this can be extended to selling). The possibility to purchase is a *power*. Indeed, the standard term is *purchasing power*.

Since $y > y' \Rightarrow B(y, p) \supset B(y', p)$, then it implies $B(y, p) \succ B(y', p)$ from the inclusion property. Hence, the question arises when prices differ.

Standard economic and statistical practice compares purchasing powers of incomes y facing prices p (hence of budget sets), and it calls *purchasing power* the number

$$\pi(y, p, b) = y/bp \tag{6}$$

where $bp = \Sigma b_i p_i$ is a (linear) price index with coefficients $b = \{b_i\} \in \Re^n_+ / \{0\}$.[11] Number π is also called *real income*.

Theorem 1 *The classical purchasing power of an income with given prices is equal to the pointed distance of the budget hyperplane of the corresponding budget set from the origin, with the vector of the coefficients of the price index as direction and measure of the distance.*

Proof Definition (6) also writes

$$\pi(y, p, b) \cdot b \cdot p = y,$$

which means from the definition of $\beta(y, p)$,

$$\pi(y, p, b) \cdot b \in \beta(y, p).$$

Therefore, from definition (5),

$$\pi(y, p, b) = P(y, p, b).$$

□

8 Equally free exchange from an equal allocation

A classical concept of equal economic freedom is full *freedom of exchange*—and hence equal such freedom in this sense—*from an equal allocation*.[12] Free exchange is an application of the basic "social liberty" ("basic rights", "negative freedom"). If each agent faces given prices, her freedom of exchange means transforming allocations x that remain on the same budget hyperplane of this agent (more generally in the corresponding budget set, but actually on the hyperplane if the agent prefers

[11] In price indexes actually used, b_i=0 for some i. Then, strictly, one can have $B(y, p) \supset B(y', p')$ and $y/bp = y'/bp'$. This relates to the choice of the $b_i > 0$ shortly discussed. In fact, the consideration of two "goods" as being the same good or not is generally more or less arbitrary. Practically, a good is chosen to represent a relevant category of goods.

[12] For instance, Kolm (1971); Thomson (2008).

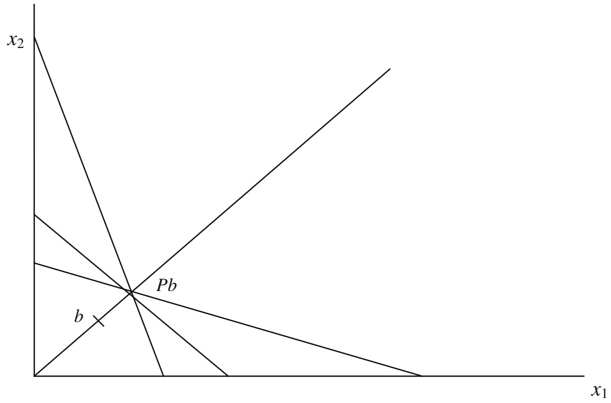

Fig. 2 Equal freedoms

larger quantities of the goods). The agents may face different sets of prices (in an important application, for instance, a price is a wage rate of an individual agent). Initially, however, the agents are given the same allocation $x^o \in \Re^n_+/\{0\}$. Then, the budget hyperplanes of the agents all pass through this point. This amounts to saying that they have the same pointed distance, and hence purchasing power, with a vector b collinear to x^o, $b = \alpha x^o$ (possibly $\alpha = 1$). If y^j and p^j denote the y and p of agent j, $P(y^j, p^j, \alpha x^o)$ is the same for all j (Fig. 2).

Theorem 2 *Equal classical purchasing power, hence equal freedom with the pointed distance ranking of budget freedom, amounts to the principle of equally free exchange from an equal allocation.*

Then, the pointed distance ordering or measure can be seen as an extension of this concept of equal economic freedom to the comparison of these freedoms when they are not equal.

9 Linear price indexes

Since there is a one-to-one correspondence between the $B(y, p)$ and the pairs (y, p), the ordering $\succsim$ also orders these pairs, with the homogeneity of degree zero property $((\lambda y, \lambda p) \sim (y, p)$ for any number $\lambda > 0)$. Prices are commonly summarized by linear price indexes $I=bp$ with $b \in \Re_+/\{0\}$. This writes the ordering $\succsim$ as an ordering of the pairs (y, bp) keeping the homogeneity of degree zero property and, therefore, as an ordering of $(y/bp, 1)$ or of y/bp, that is, of the corresponding "purchasing power".

One may also consider an ordinal *freedom function* $F(D)$ representing the freedom ordering $\succsim$. For $D \in \tilde{B}$, F writes $F(y, p)$, a function homogeneous of degree zero and increasing in y. Prices are commonly summarized by a linear index $I = bp$, homogeneous in p, and hence one can write

$$F\,(y, p) = \Phi\,(y, bp) = \Phi\,(y/bp,\ 1) = f\,(y/bp)$$

by definition of functions Φ and f. Since F is increasing in y, so is Φ and f is increasing. Hence y/bp is a specification of the ordinal function f.

Therefore, the only element introduced for ranking the freedom of choice offered by budget sets by the purchasing power *y/bp* is the aggregation of prices by a linear price index. Conversely, ranking the budget sets $B(y, p)$ by *y/bp* and hence by the pointed distance implies such an aggregation of the prices.

Theorem 3 *Budget sets are ranked by the classical purchasing power and hence by the pointed distance rule if and only if prices are summarized by a linear price index.*

Linear price indexes are neutral with respect to the units of the goods since each term $b_i p_i$ is, and in general are not symmetrical (permutable) in prices—in contrast with the case of the volume ranking. The overall measure or index is neutral with respect to the units of both the goods and the unit of account, as also shown directly by

$$P(y, p, b) = y/bp = [\Sigma\,(b_i/a_i)]^{-1}\,.$$

Almost all price indexes practically used have the linear form (like the classical Paasche and Laspeyre indexes to begin with). This is because of their meaning as the value of the bundle of goods b. This form has the correct neutrality properties. The corresponding pointed distance rule or measure does not have, for budget sets, a number of noted shortcomings it can have for general domains D. It considers a point of the undominated set, the budget hyperplane, only, $Pb \in \beta(y, p)$. The shortcoming, of course, is that it considers points of $l(b) \cap B(y, p)$ only. This is the cost of reducing sets $B(y, p)$ defined by n independent parameters, such as a or q, to a one-dimensional comparison. However, the choice of the proper index, that is, of the set of coefficients or reference bundle b, is the object of reflected consideration by statisticians, economists and other people concerned. The weights b represent sometimes an actual consumption of the situations compared (various times or countries, for instance), or an average. The choice of index is the object of private contracts and political choices, decisions and compromises when payments use this index. The weights b_i often represent a consumption norm, with debates about which good to introduce with which level b_i. The issue depends on the specific case. The choice of non-arbitrary b is usually limited. In the application to income distributions providing equal economic freedom with different wage rates, the equivalent of b is a crucial property of overall distributive justice rich in social meanings (Sect. 12.2). For a given problem, one can compare with several b, as it is commonly done. One can consider the b of a simplex $\Sigma b_i = 1$, and the domains of this simplex that provide the same rankings.

The alternative in price index theory consists in ranking $B(y, p)$ by the utility it can yield, that is by $v(y, p)$ if v is the indirect or Roy form of the utility function. Then one can take $F = v$ as freedom function. However, if the social unit under consideration is a society (e.g., a country), one first has to define a corresponding social welfare function. Moreover, this is the utmost use of utility for defining freedom, an approach strongly criticized by Pattanaik and Xu (2000).[13]

[13] However, choosing b_i by reference to some actual consumption depends on individual choices and hence on motives for these choices.

10 A linear freedom index of normalized prices

The pointed distance or purchasing power writes

$$P(y, p) = y/bp = 1/bq.$$

Hence, the corresponding ranking is equivalent to an inverse ranking by bq.

With pointed distance ranking for a given $b \geq 0$ and the definition of the functions,

$$F(y, p) \equiv G(q) = \Phi(y, bp) \equiv \Phi(1, bq) \equiv g(bq) \tag{7}$$

with a decreasing function g. Hence, with differentiable functions and $G_i = \partial G/\partial q_i$, this ranking implies, for all q,

$$G_i(q) = b_i \cdot g'(bq) \quad \text{for all } i.$$

Therefore

$$G_i(q)/G_j(q) = b_i/b_j$$

is constant for all q for each pair i, j with $b_j \neq 0$, and

$$G(q) = G(q') \Rightarrow bq = bq' \Rightarrow G_i(q) = G_i(q') \qquad \text{for all} \quad i.$$

Conversely, (1) If $G(q)$ has a linear specification bq with a constant vector b, the ranking it represents is inverse to $1/bq = y/bp$. (2) If $G_i(q)/G_j(q) = \alpha_{ij}$ is constant for all q, for each pair i, j for which $G_j \neq 0$, $G(q)$ has form $G(q) = g(bq)$ with a constant vector b. (3) Our forthcoming proof of theorem 5 implies that if

$$G(q) = G(q') \Rightarrow G_i(q) = G_i(q') \tag{8}$$

for all i, then $G = g(bq)$ with a constant vector b. But this can be seen directly differentially. If this condition holds, one has, for any constant γ, $G(q) = \gamma \Rightarrow b(\gamma)\cdot q = 1$ for an admissible vector $b(\gamma)$—this iso-G set of q is a linear manifold. Moreover, relation (8) implies that these linear manifolds are parallel, i.e., $b(\gamma)$ is the same b for all γ. Hence $G = g(bq)$.

Theorem 4 *The pointed-distance freedom ranking principle holds if and only if the freedom function in normalized prices has any of the following properties:*

(1) *a linear possible specification,*
(2) *constant rates of substitution,*
(3) *constant derivatives for the same value of this function.*

11 Miyagishima's theory

This leads to a justification of the pointed distance rule proposed by Miyagishima (2009). The ranking of two budget sets is assumed not to change when, for Xu, the a_i are multiplied by the same number, and for Miyagishima, in contrast, their inverse q_i are added the same number. Note that what is added is not quantities but prices, more exactly normalized prices, which are the inverse of quantities.

Theorem 5 (Miyagishima 2009) *The freedom ranking of budget sets is according to pointed distance, classical purchasing power or real income if and only if it does not change when given amounts are added to the normalized prices.*

That is, for any q, q' and $r \in \Re^n$ such that $q, q', q+r, q'+r \in \Re^n_{++}$,

$$G\,(q) \geq G(q') \Leftrightarrow G(q+r) \geq G(q'+r) \tag{9}$$

if and only if $G(q) = g(bq)$.

Proof This theorem results straightforwardly from Xu's proof leading to a function representing the ordering of form (4) by the *formal* change of varibles $q_i = Log\ a_i, r_i = Log\ \lambda_i$. A direct proof is the following (which therefore also constitutes a very short proof of Xu's theorem by this change of variables). Since G is a decreasing function from the inclusion relation, such a g is decreasing if $b_i > 0$ for one i. Hence such a $G = g$ satisfies obviously the property since $b \cdot (q+r) = bq + br$ and hence

$$bq \gtreqless bq' \Leftrightarrow b \cdot (q+r) \gtreqless b \cdot (q'+r).$$

Conversely, condition (9) implies the same with reverse inequalities ($\leq$) and therefore with equality:

$$G\,(q) = G(q') \Leftrightarrow G(q+r) = G(q'+r). \tag{10}$$

Then choose any two different such q and q' and any $r \neq 0$ such that $G(q+r) = G(q)$. Then, from (10), $G(q'+r) = G(q) = G(q')$. Therefore, the iso-G manifolds in $\Re^n$ are such that if one contains q, q' and $q+r$, it also contains $q'+r$. This characterizes linear manifolds. Moreover, relation (10) shows that these manifolds derive from one another by translation, that is, these hyperplanes are parallel. Therefore, G is of the form $G(q) = g(bq)$ with some constant vector b. □

12 Applications

12.1 General

For a given vector-base b, the pointed distance provides an ordering of the freedoms (or purchasing powers) offered by budget sets. It may even permit comparisons more specific than orderings since it is a number of bundles b (including fractions) that can be acquired. If this is more than comparing incomes, it applies to situations with different (non proportional) vector prices for comparable goods. These situations can refer to countries or regions, dates, various economic entities, effects of discriminations or of policies, and individuals notably because they have different wage rates (price of labour or of leisure). The ordering suffices to define equal liberty, which corresponds to budget hyperplanes passing through the same point (they constitute a "pencil" of hyperplanes). Considering the ranking only, the pointed distances for different entities are comparable by more or less, and one can define the minimum. Comparing several cases for a given set of entities, one can define and consider, for freedoms, maximin, leximin, Pareto-like comparisons, such comparisons after permutation of the distances in one state and notably first-order stochastic dominance, number of distances larger or smaller than a given one, etc. Considering the distances as meaningful quantities permits all operations that can be performed with incomes, including all concepts of comparing or measuring inequalities in liberty.

12.2 Macrojustice as equal economic freedom

12.2.1 The theory

On ethical grounds, equal freedom of choice should be sought when freedom of choice is the only direct value of the relevant conception of justice. This value is freedom of choice rather than the chosen item or the utility it provides when the choosing individuals are deemed accountable for their personal preferences or utilities and for their choice given their freedom of choice provided by their domains of possibilities. This accountability of their choice is usually justified by their responsibility. Moreover, the direct values exclude preferences or comparing them in a number of cases. One of these cases is particularly important: this is the common opinion about the overall distributive justice (in "macrojustice") implemented by the main fiscal tools such as the income tax. Indeed, people do not usually think that someone should pay a higher income tax than someone else because the other is presumably more able to enjoy the euros taken away or less able to enjoy the euros left (this discards both utilitarian and egalitarian welfarist conceptions, and undoubtedly all intermediate ones). One possible interpretation of this fact—retained, for instance, by Rawls (1971)—is that freedom of choice is the direct value of justice. This implies that it should ideally be equal among the individuals, from a reason of rationality in the basic sense of providing a reason, since there is no relevant item which could justify non-equal solutions.[14]

However, the comparison of freedoms obtained above is non-trivial only when prices differ between the compared cases. For individuals and perfect competition, this happens when the wage rates are among the prices, and hence leisure and labour are among the goods. This is precisely the case for the determination of the just disposable income, just noted. The issue then is equal freedom of choice—notably of labour and income—and the result consists in the corresponding optimum transfers.

[14] This logical necessity of *prima facie* or ideal equality in the relevant items is fully explained notably in Kolm 2010 or 1997 (translation of 1971), foreword, Sect. 5 (see also 1996a, Chap. 2 and 2004a).

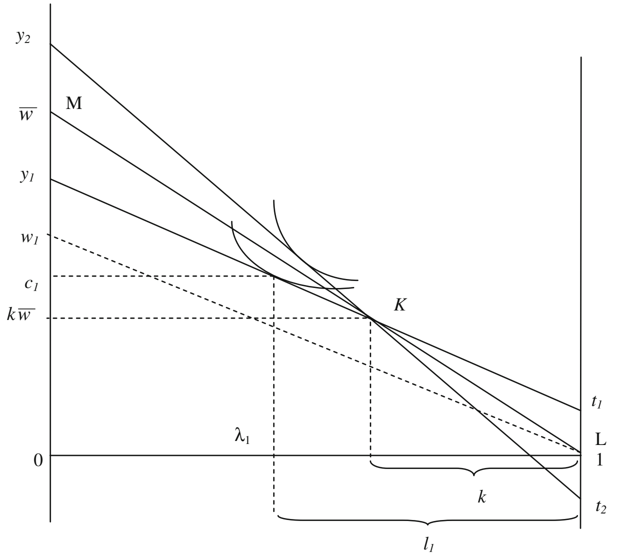

Fig. 3 Equal freedom for macrojustice

Consider m individuals indexed by i. There are two goods, income which enables one to buy consumption goods, and leisure or labour. Individual i has income c_i, leisure λ_i, labour l_i, with $\lambda_i + l_i = 1$ by choice of the unit of time, and given productivity and wage rate w_i. Her earned income is $w_i \lambda_i$. She can receive transfer t_i, which is a tax of $-t_i$ if $t_i < 0$.

Individual i's income is

$$c_i = w_i l_i + t_i.$$

Her *total income*, including the value of leisure λ_i at its market price w_i, is

$$y_i = c_i + w_i \lambda_i = w_i + t_i$$

and corresponds to the income of the previous sections. This is the equation of individual i's budget line in the space of leisure λ_i and income c_i (Fig. 3).

From the above, equal budget freedom implies that all budget lines pass through the same point.[15] Denote as k and η the labour and the income corresponding to this point, respectively (the leisure is $1 - k$). We have, for all i,

$$\eta = w_i k + t_i.$$

Hence,

$$m\eta = k\Sigma w_i + \Sigma t_i.$$

15 This is an axiom in Maniquet (1998).

If this distribution is financially balanced, $\Sigma t_i = 0$ (e.g., Musgrave's "distribution branch"). Then, $\eta = k\overline{w}$ where $\overline{w} = (1/m)\,\Sigma w_i$ is average wage rate, and $t_i = k \cdot (\overline{w} - w_i)$.

The t_i constitute a redistribution which amounts to redistributing equally the product of the same labour k of all individuals (with their different productivities). This is *equal-labour income equalization* (ELIE). It produces *equal pay for equal work* for the *equalization labour* k. It also amounts to the grant of an *equal universal basic income* (of $k\overline{w}$) to everyone, *financed by an equal sacrifice* of each in terms of labour (k) or according to capacities w_i—since each individual i then pays $w_i k$.[16] Equivalently, this distribution amounts to each individual transferring to each other the proceeds of the same labour of hers (k/m)—this is *general balanced labour reciprocity*. The final outcome also amounts to all individuals *freely choosing* their labour and earnings *from an equal allocation* of both leisure $1 - k$ and income $k\overline{w}$. Moreover, $y_i = w_i + t_i = k\overline{w} + (1 - k)\,w_i$ shows that the operation is a *concentration* of the total incomes (a linear uniform concentration towards the mean) from their values $y_i = w_i$ for $k = 0$; this is the structure that most uncontroversially reduces inequality.[17]

Since human capacities provide an extremely large part of the economic value of the natural resources (labour provides the largest part of national income and it also largely created capital), this solution allocates the bulk of resources. It constitutes overall distributive justice, a part of macrojustice along with social freedom. The case $k = 0$ is classical full self-ownership. The equalization labour k is a coefficient of equalization, redistribution, community of human resources and reciprocity, and a minimum income in so far as individuals are not responsible for their low income ($c_i > k\overline{w}$ if $l_i > k$, and $c_i = k\overline{w}$ if $w_i = 0$).[18] The amounts presently redistributed by national policies are those that would correspond to an equalization labour of one to 2 days per week. In standard application, this distributive scheme is applied to individuals i with $l_i \geq k$ only. However, other cases can be reduced to this one by specific theories or devices (unemployment, part-time labour contracts, etc.). One can then show that an individual's interest is to work with her most productive capacities – and thus to reveal them.

One has

$$c_i = \overline{w}k + w_i \cdot (l_i - k)$$

which shows that, with normal labour $l_i \geq k$, income c_i is the sum of an egalitarian income $\overline{w}k$—the same for all, produced by the same labour k –, and of a "classical liberal" income $(l_i - k)w_i$ which is the earnings of individual i with her wage rate w_i and her freely chosen labour $l_i - k$. This result amounts to individuals receiving

16 Van Parijs (1995) calls a universal basic income for all equal "real freedom". This restricts real freedom to the individual income for $\ell_i = 0$, not counting the possibilities provided by working with a wage rate smaller or larger. This is freedom from the necessity to work only. Moreover, this basic income has to be financed somehow (van Parijs proposes to finance it by a tax on earned income).

17 Cf. Kolm (1999b).

18 Approximately, $k\overline{w}$ is to average earnings as k is to average labour duration.

according to their *work* (*desert*) for the equalization labour k, and according to their *works* due to their *work and capacities* (*merit*) for the rest of their labour.

A national tax law has solved the information question (about the w_i) by exempting overtime labour earnings from the income tax, over a rather low benchmark, with actually no cheating.[19]

The prices are 1 for income and w for leisure-labour (i.e., w_i for individual i). Since the common possible allocation is $k\overline{w}$ for income with k for labour ($1-k$ for leisure), the price index can be taken as $k\overline{w} + (1-k)w = y$ (y_i for individual i, her total income). Its choice amounts to the choice of the equalization labour or coefficient k, an issue which has been analyzed in depth.[20]

If, more generally, individuals' earnings are not necessarily linear (scale effect, imperfect labour market, etc.), they are the increasing function $f_i(l_i)$ for individual i whose domain of choice then is $c_i \leq f_i(l_i) + t_i$. The pointed distance freedom ranking can also apply, providing, for equality, the same $c_i = \gamma = f_i(k) + t_i$ for some equal labour k. Then, with $\Sigma t_i = 0$, $\gamma = \overline{f}(k) = (1/m)\Sigma f_i(k)$, and therefore $t_i = \overline{f}(k) - f_i(k)$ and individual i chooses

$$c_i = f_i(l_i) - f_i(k) + \overline{f}(k).$$

This is again an equal-labour income equalization for labour k, and a basic income $\overline{f}(k)$ financed by an equal labour of all. This theory extends to multidimensional labour (duration, training, effort, etc.) and to the case of various constraints such as partial or total involuntary unemployment).

12.2.2 Equal freedom and classical liberalism

The obtained results have important implications for the conception (the philosophy) of economic liberty. A basic structure of this conception is the division between social liberty (negative, civic, basic rights, including free exchange without third party forceful interference) and real freedom which adds the agent's (other) means. Their conjunction is the agent's total freedom. The pointed distance (purchasing power) ranking of economic freedom gives, as equal freedom, the classical free exchange from an equal allocation. This holds in particular for the application to income distribution. In this case, however, the initial allocation results, when $k > 0$, from a partial redistribution of the value of individuals' capacities. Now classical liberalism—the founding

[19] The volume Kolm (2004a) provides the answers to the questions concerning : intensity of labour and effort; education and training; multidimensional labour; non-linear production functions of labour; involuntary unemployment; information about wage rates and given productivities; determination of the equalization labour and coefficient k; comparison with the relevant policies, proposals and philosophies; method of practical implementation by reform of the present fiscal distributive tools (one of this proposals inspired the present French tax law of exempting overtime labour earnings from the income tax over a low benchmark, which shows a way of basing the tax on wage rates, actually without cheating because this could not escape the tax administration's notice—the education input of labour is taken care of by free public education financed by this tax), etc. Other aspects of this policy are analyzed in the collective volume edited by Gamel and Lubrano (2010).

[20] Part 4 of the previous reference.

social ethics of the modern world—defines itself as advocating social liberty and in particular free exchange, and full self-ownership, it deems these two principles to be equivalent, and it sees them as forbidding distributive taxation. If $k = 0$, hence $t_i = 0$ for all i, this fits with this conception. If $k > 0$, however, a new situation obtains. It respects social freedom and unfettered free exchange, but "after" some distribution of the given resources performed by the transfers t_i. This leaves intact the use-right (the right to use) in one's own capacities, but redistributes, to some extent, the economic value of this right, that is the rent of these capacities. This is another conceptual way of applying social liberty, consistent with redistribution. Classical liberalism then is the particular case $k = 0$. With $k > 0$, however, there are taxes ($t_i < 0$) paid by individuals with $w_i > \overline{w}$, which appear as liabilities in the basic ("initial") distribution. Classical liberalism commonly objects to taxation on the grounds that it forces people to work more (or to consume less), which would be an infringement of their freedom. With ELIE with $k > 0$, however, individuals with $w_i < \overline{w}$ receive a subsidy which enables them to consume more or work less. For the other people, it is worth noting that someone who pays a higher tax than someone else (who may pay no tax) has, de facto, a higher freedom of choice: she can both work less and consume more. Indeed, the tax $-t_i = k \cdot (w_i - \overline{w})$ is higher when w_i is higher, and

$$c_i = k\overline{w} + (l_i - k)w_i$$

shows, with the normal $l_i > k$, that a higher w_i gives more freedom—it permits to work less and/or consume more. A consequence is that classical liberalism is on safer grounds by focusing on its other justification, full self-ownership, based on some concept of "natural right".

13 Freedom and utility

13.1 A question

A second philosophical issue is whether a ranking or measure of freedom should be independent from any preference of the agent. This is a central concern for Xu in the work discussed here and in previous ones with Pattanaik (Pattanaik and Xu 2000). This separation seems a priori an interesting property for consideration. It is respected by the pointed distance rule obtained (with possible qualification for the choice of parameters b). However, it does not seem possible to condemn all uses of the term freedom that do not respect it. Note that prices, which determine budget sets, depend in general on supply and demand, and hence on preferences (perhaps not on those of a specific small agent). Moreover, features of preferences that all possible agents would have may be acceptable. For instance, for desired goods, choices would be restricted to the undominated set even if there is not free disposal (which would not be used anyway). And the corresponding surface ranking differs from the volume ranking (the pointed distance ranking or measure raises no such dilemma). Moreover, other uses of preferences are not a priori illegitimate. The main ones can be classified in two categories.[21]

13.2 Equivalent or potential freedom

13.2.1 General

In a group of cases preferences or utility are not an end-value but are used to describe the agent's behaviour or potential behaviour. This occurs notably when the ethical end-value is freedom but one may consider the agents' potential freedom, a freedom with which agents would have chosen their actual allocation and situation. Then, these allocations or situations are appraised with the final value being this liberty. Morally valuing the outcome of a hypothetical liberty is classical since it is the case of one of the most important theory of social ethics, the social contract (for free exchange or agreement).

Given an agent with an allocation or situation x and a utility function $u(\xi)$, domains of possible choice D such that

$$x = \arg\max u(\xi)/\xi \in D$$

are said to be *freedom-wise equivalent* to x and between themselves. Then, the evaluation of x is considered as equivalent to that of any of these D. This implies that liberty is valued for what it permits the agent to obtain only. The domains D can have various natures, and those of various agents can be compared in various ways (such as identity, inclusion, pointed distance, etc.). This gives rise notably to the two following types of application. The indexes qualifying x, u, D, etc. denote the agents.

13.2.2 Equivalent or potential budget sets

Consider $D \in \tilde{B}$ (hence $D \subset \Re^n_+$), $D = B(y, p)$ with income y and the price vector p. Then $x, \xi \in \Re^n_+$. The *equivalent (or potential) budget set* of an agent with allocation x and differentiable utility function $u(\xi)$ is the budget set from which she chooses or would choose x. The corresponding income y and price vector p are her equivalent or potential income and prices, denoted as $y(x)$ and $p(x)$. One has

$$x = \arg\max u(\xi)/\xi \in B[y(x), p(x)]$$

and $B[y(x), p(x)]$ is the equivalent or potential budget set. If $v(y, p)$ denotes the agent's Roy indirect utility function, one has $v[y(x), p(x)] = u(x)$ (see Fig. 4).

Equivalent or potential budget sets can be ranked or measured by their purchasing power or pointed distance, which ipso facto ranks the agents actual allocations x. This

[21] Most of these concepts or results also apply if "utility" means some output, the goods are the inputs that permit to produce it and the utility function is the corresponding production function.

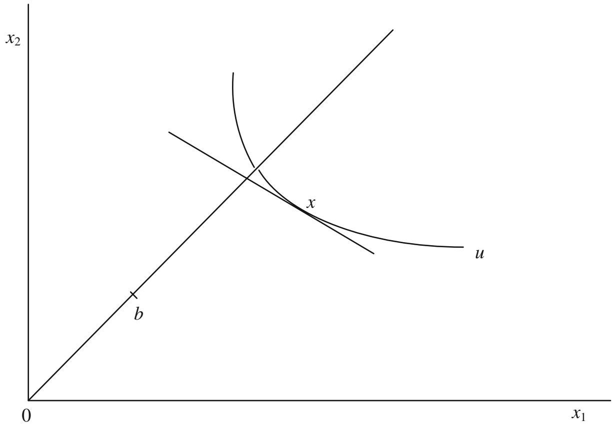

Fig. 4 Equivalent budget and utility-freedom

measure is $P[y(x), p(x), b]$ for some $b \in \Re^n_+/\{0\}$. With several agents all the previously noted uses of this ranking or measure can be used. In particular, there may be equality. The optimum income distribution and taxation of Sect. 12.2 is the particular case in which $x = (\lambda, c)$, the pair of leisure (or labour $l = 1 - \lambda$) and income that buys consumption goods.

13.2.3 Equity-no-envy as equal potential freedom and generalization

If x^i and $u^i(x^i)$ are an allocation and a utility function of individual i, for several individuals i, j, etc., a classical equity principle is *equity-no-envy* defined as,

$$u^i(x^i) \geq u^i(x^j) \quad \text{for all } i, j. \tag{11}$$

This does not describe the sentiment of "envy" proper or "strong envy" since individual i's such envy of individual j for her allocation x^j would have to be described with a utility function $U^i(x^i, x^j)$. However, the full economic theory of envy nevertheless uses this criterion for "envy-free preferences".[22] Moreover, the most important property of this principle is that it is, in fact, a concept of equal potential freedom. An actual freedom is considered here as a particular case of a potential freedom.

Definition An overall allocation $\{x^i\}$ satisfies *equal potential freedom* when each individual allocation x^i can be chosen by individual i from an identical domain of choice D, the same for all:

$$\exists D : \forall i, \quad x^i = \arg\max u^i(\xi)/\xi \in D. \tag{12}$$

Theorem 6 [23] *Equity-no-envy and equal potential freedom are identical.*

22 This would be $V^i(x^i) = U^i(x^i, x^i)$. See Kolm (1995).

23 Kolm (1971).

Proof Clearly, (12) implies (11). Conversely, if (11) holds, $D=\{x^i\}$, the set of individual allocations, satisfies condition (12). Other domains D satisfying (12) are obtained by adding any individual allocation ξ such that $u^i(\xi) \leq u^i(x^i)$ for all i. □

This result extends to a comparison of individuals' allocation by more or less potential freedom defined by the inclusion of equivalent domains of choice. This permits, in particular, to replace the cases in which equity-no-envy is not Pareto-efficient by a Pareto-efficient maximin in potential freedom.[24]

13.3 Utility-freedom

With potential freedom, utility or preferences are only a mean to relate allocation and potential freedom. In contrast, freedom of choice is often provided by capacity or power to obtain (as with purchasing power), and one may consider, in particular, an agent's capacity or power (and hence freedom) to obtain some level of "utility". A domain of choice D provides an individual having utility function $u(x)$ with the possibility to reach at most utility $U(D) = \max u(\xi)/\xi \in D$. With $x \in \Re^n_+$ and $\tilde{B} \ni D = B(y, p)$, $U(D) = v(y, p)$ where v is the individual's indirect (Roy) utility function. The "utility-freedom" provided by budget sets $B(y, p)$ has as freedom function $F(y, p) = v(y, p)$.

With $\xi \in \Re^n_+$ and a direct utility function $u(\xi)$, the utility levels $\tilde{u}$ can be specified by a pointed distance $P(u, \tilde{u}, b)$ defined as

$$u[P(u, \tilde{u}, b) \cdot b] = \tilde{u}$$

for some vector $b \in \Re^n_+/\{0\}$, that is, the distance from the origin of the intersect of $l(b)$ with the iso-utility hyper-surface $u(\xi) = \tilde{u}$, with the length of vector b as unit. This b-pointed-distance utility of allocation x for an individual with utility function $u(\xi)$ is $P[u, u(x), b]$ (Fig. 4). This can rank the utilities that individuals i with utility functions $u^i(\xi)$ can reach with their allocations x^i. The case of equality for several individuals is the principle of *egalitarian equivalence* (Pazner and Schmeidler 1978) of the overall allocation $\{x^i\}$. Then, $P[u^i, u^i(x^i), b] = Q$ are the same for all i and the allocation Qb is the *equal equivalent* allocation. Egalitarian equivalence is consistent with Pareto efficiency for the allocation of a set of quantities of goods $\Sigma x^i \leq X \in \Re^n_+$, as shown by Pazner and Schmeidler. One can show, differently, that, for such an allocation and more generally for $\Sigma x^i \in \Pi \subset \Re^n_+$ (i.e., with possibilities of transformation of the goods or production), there even is one Pareto-efficient egalitarian-equivalent overall allocation with the equal equivalent on the line $l(b)$ *for each vector b*.[25]

24 See Kolm (1999a).

25 Kolm (1996a) and the references noted there.

14 Ordering invariance and generalization: additive separability

We have seen that there are two ways for the ordering of a property characterized by a number of variables to be invariant under a defined type of change of the variables. Let Z denote this set of variables and $\Psi(Z)$ a function that represents this ordering. Let Z incur such a transformation and becomes $\tilde{Z}$.

With *ordering invariance*, function Ψ does not change and $\Psi(Z) \geq \Psi(Z') \Rightarrow \Psi(\tilde{Z}) \geq \Psi(\tilde{Z}')$ for all relevant Z and Z'. This implies a priori a particular structure of function Ψ and hence of the ordering.

With *property invariance*, function Ψ incurs a contravariant transformation and becomes function $\tilde{\Psi}$ such that $\tilde{\Psi}(\tilde{Z}) = \Psi(Z)$ for all relevant Z. Then the ordering is trivially maintained.

Xu's and Miyagishima's theories for budget sets belong to ordering invariance. Formally, Xu's scale invariance is just a change of variables: each is multiplied by a constant. As we have noted, the economic meaning of a change in measuring units leads, rather, to the most classical case of property invariance.

Let us consider now ordering invariance only. For budget sets, $G(q)$ is the ordering function with the normalized prices $q = p/y = \{1/a_i\}$. For characterizing specifications of ordinal functions (here G), denote as $\sim$ the corresponding relation "equal up to an increasing transformation". Scale invariance and the volume ranking correspond to a product $G \sim \Pi q_i = (\Pi a_i)^{-1}$. Invariance to addition and the pointed distance ranking correspond to a linear $G \sim \Sigma b_i q_i$. These two results are particular cases of the following one.

Theorem 7 *The ordering function is additively separable as* $G \sim \Sigma\beta_i f_i(q_i)$ *if and only if the ordering does not change if a variable* q_i *is augmented by* $\varepsilon_i g_i(q_i)$ *with* $\varepsilon_i \to 0$ *and* $g_i f_i' = c_i$*, a constant.*

Proof Change the variables into $z_i = f_i(q_i)$. Then, z_i is added the constant ε_i' if and only if q_i is added $\varepsilon_i g_i(q_i)$, with $g_i f_i' = c_i$ and $\varepsilon_i' = c_i\varepsilon_i$, since z_i becomes

$$z_i + f_i'(q_i) \quad \varepsilon_i g_i(q_i) = z_i + c_i\varepsilon_i.$$

Applying theorem 6 with the variables z_i then shows that if and only if this change does not affect the ordering, $G \sim \Sigma\beta_i z_i = \Sigma\beta_i f_i(q_i)$ with any set of constant $\beta_i > 0$. □

The two polar cases are scale invariance and a logarithmic f_i, and invariance to addition and a linear f_i. In the former case, $f_i = \gamma_i \mathrm{Log}\ q_i$ (γ_i is constant) and $g_i = c_i\gamma_i^{-1}q_i$, and the transformation amounts to multiplying q_i by $1 + \varepsilon_i c_i \gamma_i^{-1}$. In the latter case, $f_i = \rho_i q_i + \sigma_i$ (ρ_i and σ_i are constant) and $g_i = c_i\rho_i^{-1}$, and the transformation is the addition of the constant $\varepsilon_i c_i \rho_i^{-1}$ to q_i.

Two particularly interesting forms of the functions f_i and g_i are those in which they are both power or exponential functions. The cases of scale invariance and of constant addition are particular cases of each of these two cases. In the power case, for instance, $g_i = \mu_i q_i^{\alpha_i}$ and $f_i = \lambda_i q_i^{1-\alpha_i}$ with $(1-\alpha_i)\lambda_i\mu_i = c_i$. Then, α_i=0 means that q_i is added a constant $\varepsilon_i\mu_i$. And $\alpha_i = 1$ means that q_i is multiplied by the constant $(1 + \varepsilon_i\mu_i)$ (scale invariance).

The functions f_i and g_i for different goods i can a priori be of different types. The case in which they are of the same type is meaningful, however. If f_i is the logarithmic $f_i = \gamma_i$ Log q_i for all i, $G \sim \Pi q_i^{\eta_i}$ with constant $\eta_i = \beta_i \gamma_i$, which is the volume ordering if $\eta_i = \eta$ for all $i((\Pi\alpha_i)^{-\eta})$. With the linear or affine $f_i = \rho_i q_i + \sigma_i$ for all i, $G \sim \Sigma\tau_i q_i$ with constant $\tau_i = \beta_i\rho_i$, which is the purchasing power or pointed distance ordering with vector $b = \{\tau_i\}$. With power functions $f_i = \lambda_i q_i^{1-\alpha_i}$ for all i, $G \sim \Sigma\pi_i q_i^{1-\alpha_i}$ with $\pi_i = \beta_i\lambda_i$. Then, if $\alpha_i = \alpha$ are the same for all i, and $\nu = (1-\alpha)^{-1}$ one also has $G \sim (\Sigma\pi_i q_i^{1/\nu})^\nu$, a function with constant elasticity of substitution which gives, as particular cases, the purchasing power or pointed distance ordering if $\nu = 1$ and the volume ranking if $\nu \to 0$ and $\pi_i = \pi$ are the same for all i.[26]

The characteristic of the cases of this section is that the changes of variables are for each variable independently from the others. Hence, philosophically, the property of additive separability of the ordering function G. Other cases may occur. For instance, in economics, when aggregating goods of each category (would it only be because the definition of a good—the choice to call several items instances of the same good—is ambiguous).

15 Concluding remarks

The general properties of comparing the freedoms offered by domains of choice, the problem of contravariance, the properties of the pointed-distance ranking of budget sets, the application to distribution and taxation in macrojustice, the general theory of ordering invariance, and the concepts and properties of utility-based liberty have been the main issues raised by the comparison of real economic freedom. All are amenable to important further developments. The last one, in particular, deserves more subtle considerations.

Actually, indeed, freedom and "utility" are still more intricately related. The loosening of a binding constraint is an increase in freedom and it permits reaching a higher satisfaction. The very fact that a person is not more satisfied than she actually is whereas she would like to be constitutes a constraint and hence a limitation of freedom. These two facts make freedom and utility perfectly correlated. For economics, there is no freedom without some utility for the agent to maximize, to be free is to be forced by utility, freedom is the dictatorship of utility. Another, deeper issue is that individuals can affect their own preferences in various ways (reflection, habit, training, etc.). To this extent, they have some freedom for the choice of their utility.[27] All this, of course, has to be qualified by the various very important aspects of the psychology (and psychosociology) of freedom: the costs of comparing and choosing, a direct interest in this activity, disliking or liking the responsibility implied by the free choice, the anguish of choice (Kierkegaard, Sartre) or the exhilaration of being free, and so on. The other vast and rich domain of freedom analysis is that of interfering

26 For other properties of all these cases, see Kolm (2004a, pp. 423–425).

27 See, for instance, Kolm (1982) and in Elster, ed. 1985, pp. 233–263.

freedoms of various agents (an issue hidden here by the assumption of given prices and by the generality of the classical definition of social liberty).

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